fiziks

Forum for CSIR-UGC JRF/NET, GATE, IIT-JAM/IISc, JEST, TIFR and GRE in PHYSICS & PHYSICAL SCIENCES

Mechanics and General Properties of Matter

(IIT-JAM/JEST/TIFR/M.Sc Entrance)

Head office
fiziks, H.No. 23, G.F, Jia Sarai,
Near IIT, Hauz Khas, New Delhi-16
Phone: 011-26865455/+91-9871145498

Branch office
Anand Institute of Mathematics,
28-B/6, Jia Sarai, Near IIT
Hauz Khas, New Delhi-16

Website: www.physicsbyfiziks.com
Email: fiziks.physics@gmail.com

MECHANICS

1. Newton's Laws of Motion……(1-41)

1.1 Newton's Laws of Motion

1.1.1 Newton's First Law

1.1.2 Newton's Second Law

1.1.3 Newton's Third Law

1.2 Application of Newton's Law of Motion

1.2.1 Free Body Diagram

1.2.2 Equation of Constrain

1.2.3 Equation of Motion

1.3 The Motion in Two Dimensional in Polar Coordinate

1.3.1 The Position Vector in Polar Coordinate

1.3.2 Velocity Vector in Polar Coordinate

1.3.3 Acceleration Vector in Polar Coordinate

1.3.4 Circular Motion

2. Energy and Momentum……(42-89)

2.1 Different Form of Energy used in Mechanics

2.1.1 Kinetic energy

2.1.2 Potential energy

2.2 The work-energy Theorem

2.3 Work Done by a Uniform Force

2.4 Non conservative forces

2.5 Energy Diagrams

2.5.1 Energy Diagram of Bounded Motion

2.5.2 Energy Diagram of Unbounded Motion

2.6 Stability and Unstability in One Dimension

2.6.1 Small Oscillation

2.7 Center of Mass

2.7.1 Center of Mass of Continuous System

2.8 Conservation of Momentum

2.8.1 Elastic Collision in One Dimension

2.8.2 Perfectly Inelastic Collision in One Dimension

2.8.3 Elastic Collision in Two Dimensions

2.9 Variable Mass and Momentum

3. Central Force and Kepler's System……(90-116)

3.1 Central Force

3.1.1 Angular Momentum and Areal Velocity

3.1.2 Total Energy of the System

3.1.3 Condition for Circular Orbit

3.1.4 Equation of Motion and Differential Equation of Orbit

3.2 Two Body Problem

3.2.1 Kepler's Problem

3.2.2 Kepler's First Law

3.2.3 Equation of Motion

3.2.4 Relationship between Energy and Eccentricity

3.2.5 Kepler's Second Law:

3.2.6 Kepler's Third Law:

4. Moment of Inertia Rotational Dynamics……(117-177)

4.1 Rigid Body Dynamics

4.1.1 Angular Momentum and Inertia Tensor

4.1.2 Principal Moment of Inertia

4.1.3 Theorem of Parallel Axes:

4.1.4 Theorem of Perpendicular Axes:

4.2 Particular Cases of Moment of Inertia

4.2.1. Moment of Inertia of a Uniform Rod

4.2.2 Moment of Inertia of a Rectangular Lamina (or bar)

4.2.3 Moment of inertia of a hoop or a thin circular ring.

4.2.4 Moment of inertia of a circular lamina or disc:

4.2.5 Moment of Inertia of an Annular Ring or Disc

4.2.6 Moment of Inertia of a Solid Cylinder:

4.2.7 Moment of Inertia of a Solid Cone

4.2.8 Moment of Inertia of a Hollow Cylinder

4.2.9Moment of inertia of a uniform hollow sphere about a Diameter

4.2.10 Moment of Inertia of a Uniform Solid Sphere about a Diameter

4.3 Rotational Dynamics

4.3.1Angular Momentum

4.3.2 Torque

4.4 Motion Involving Both Translation and Rotation

4.4.1 Angular Momentum of a Rolling Wheel

4.4.2 The Torque on a Body

4.4.3 Condition of Rolling Without Slipping

5. Fluid Mechanics……(178-213)

5.1 Pressure in a fluid

5.1.1 Variation of Pressure with Height

5.1.2 Pascal's Law

5.1.3 Atmospheric Pressure and Barometer

5.1.4 Manometer

5.2 Archimedes' Principle

5.2.1 Pressure Difference and Buoyant Force in Accelerating Fluids

5.2.2 A Liquid Placed in an Elevator

5.2.3 Free Surface of a Liquid in Horizontal Acceleration

5.3 Steady and Turbulent Flow

5.3.1 Line of Flow: Streamline

5.3.2 Irrotational flow of an incompressible and non-viscous fluid

5.4 Equation of Continuity

5.5 Bernoulli's Equation

1. Newton's Laws of Motion

1.1 Newton's laws of motion are three physical laws that, together, laid the foundation for classical mechanics. They describe the relationship between a body and the forces acting upon it, and its motion in response to those forces.

1.1.1 First Law: When viewed in an inertial reference frame, an object either remains at rest or continues to move at a constant velocity, unless acted upon by an external force.

The first law states that if the net force (the vector sum of all forces acting on an object) is zero, then the velocity of the object is constant. Velocity is a vector quantity which expresses both the object's speed and the direction of its motion; therefore, the statement that the object's velocity is constant is a statement that both its speed and the direction of its motion are constant. The first law can be stated mathematically as $\sum \vec{F} = 0 \Rightarrow d\vec{v}/dt = 0$.

Consequently,

- An object that is at rest will stay at rest unless an external force acts upon it.
- An object that is in motion will not change its velocity unless an external force acts upon it.

Newton placed the first law of motion to establish frames of reference for which the other laws are applicable. The first law of motion postulates the existence of at least one frame of reference called a Newtonian or inertial reference frame, relative to which the motion of a particle not subject to forces is a straight line at a constant speed. Newton's first law is often referred to as the *law of inertia*. Thus, a condition necessary for the uniform motion of a particle relative to an inertial reference frame is that the total net force acting on it is zero.

1.1.2 Second Law: The vector sum of the external forces $\vec{F}$ on an object is equal to the mass m of that object multiplied by the acceleration vector $\vec{a}$ of the object $\vec{F} = m\vec{a}$. The second law states that the net force on an object is equal to the rate of change (that is, the *derivative*) of its linear momentum $\vec{p}$ in an inertial reference frame

$$\vec{F} = \frac{d\vec{P}}{dt} = \frac{d(m\vec{v})}{dt}$$

The second law can also be stated in terms of an object's acceleration. Since Newton's second law is only valid for constant-mass systems, mass can be taken outside the differentiation operator by the constant factor rule in differentiation. Thus,

$$\vec{F} = m\frac{d\vec{v}}{dt} = m\vec{a}$$

where $\vec{F}$ is the net force applied, m is the mass of the body, and $\vec{a}$ is the body's acceleration. Thus, the net force applied to a body produces a proportional acceleration. In other words, if a body is accelerating, then there is a force on it.

Consistent with the first law, the time derivative of the momentum is non-zero when the momentum changes direction, even if there is no change in its magnitude; such is the case with uniform circular motion. The relationship also implies the conservation of momentum: when the net force on the body is zero, the momentum of the body is constant. Any net force is equal to the rate of change of the momentum.

Any mass that is gained or lost by the system will cause a change in momentum that is not the result of an external force. A different equation is necessary for variable-mass systems.

Newton's second law requires modification if the effects of special relativity are to be taken into account, because at high speeds the approximation that momentum is the product of rest mass and velocity is not accurate.

1.1.3 Third Law: When one body exerts a force on a second body, the second body simultaneously exerts a force equal in magnitude and opposite in direction on the first body.

An impulse $\vec{J}$ occurs when a force $\vec{F}$ acts over an interval of time Δt, and it is given by

$$\vec{J} = \int \vec{F}.dt$$

Since force is the time derivative of momentum, it follows that

$$\vec{J} = \Delta\vec{p} = m\Delta\vec{v}$$

This relation between impulse and momentum is closer to Newton's wording of the second law.

1.2 Application of Newton's Law of Motion

To apply Newton's law of motion one should follow following step .

Step 1- Draw free body diagram and identified external forces

Step 2 – Write down equation of constrain.

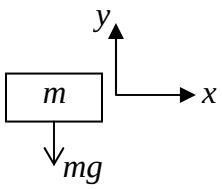
Step 3 – Write down Newton's law of motion.

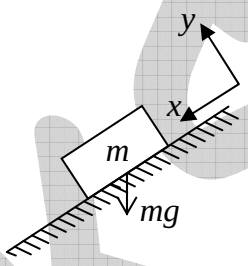
1.2.1 Free Body Diagram

To draw free body diagram assume body is point mass of mass m then assume coordinate axis as x-axis is direction of motion .the perpendicular to plane of motion is y-axis , after determining the axis find all type of external forces.

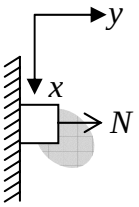
Some of common external forces are discuss below.

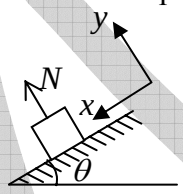
1. Weight: The weight of body is $w = mg$. The weight of the body is always in vertically downward direction, as shown in figure.

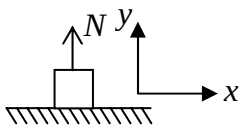




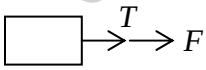
2. Normal forces when body of mass touching any surface then surface will exert normal force N .The direction of normal force is perpendicular to plane of surface

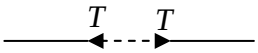




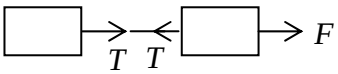


3. Tension: Force F is exerted on mass m through string .Then any section of string is pulled by two equal and opposite forces. Any one of these forces is called tension . Tension always gives pulling effect. In the figure, the force F is acted on mass m through string so there is tension T in the string giving pulling effect.





Now let us analyze another system in which two masses of mass m_1 and m_2 attached with string and force F acted on mass m_1 as shown in fig .Due to force F tension T developed in string if one will draw free body diagram for m_1 the tension will act towards left and if free body diagram draw for m_2 the tension will act towards right . In both the case tension will produce pulling effect on mass m_1 and m_2.



Same thread same tension
Different thread different tension

If the rope is mass less the tension in string is same for all sections of string other wise it will depend on mass of segment of the rope.

4. Friction: in general friction force is force which is responsible to oppose the motion .

There is two type of frictional force

a) **Static Friction**: when there is not any relative motion between surface and body m then frictional identified as static frictional which is force is equal to external force acted on mass m .the direction of frictional force is tangent or parallel to surface . if μ is coefficient of friction between surface and mass m then maximum value of frictional force is μN .

In summary we take the force of friction f to behave as follows

$0 \le f \le \mu N$ where f oppose the motion that would occur in its absence

b) **Kinetic Friction**: if there is relative speed between surface and mass m then frictional force is identifying as $f_k = \mu N$ where μ is coefficient of kinetic friction and N is normal force . the direction of f is directed opposite to the motion .

Example: Block and Wedge with Friction

A block of mass m rests on a fixed wedge of angle θ. The coefficient of friction is μ. (For wooden blocks, μ is of the order of 0.2 to 0.5). Find the value of θ at which the block starts to slide.

Solution: In the absence of friction. The block would slide down the plane; hence the friction force f points up the plane. With the coordinate shown, we have

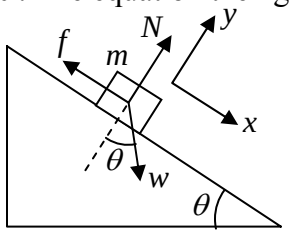
$$m\ddot{x} = W\sin\theta - f \quad \text{and} \quad m\ddot{y} = N - W\cos\theta = 0$$

when sliding starts. f has its maximum value μN , and $\ddot{x} = 0$. The equation then give

$$W\sin\theta_{\max} = \mu N$$

$$W\cos\theta_{\max} = N$$

Hence, $\tan\theta_{\max} = \mu$



Notice that as the wedge angle is gradually increased from zero, the friction force grows in magnitude from zero toward its maximum value μN since before the block begins to slide we have $f = W\sin\theta \qquad \theta \le \theta_{\max}$

5. Pseudo Force: some time the observer is on accelerated frame of reference then observer will measured acceleration without any external force, which invalidate newtons law of motion. So one can use a concept of pseudo force . if a_0 is acceleration of observer and he is measuring the pseudo force F_s on rest mass m. The magnitude of pseudo $F_s = ma_o$ and direction is opposite to direction of observer.

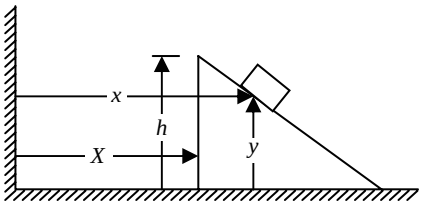
One can understand the concept of pseudo force from following example

Assume two observer A and B moving with acceleration $a_A\hat{i}$ and $-a_B\hat{i}$ respectively with respect to mass m then observer A measure pseudo force ma_A is $-\hat{i}$ direction and observer B measure pseudo force ma_B is $\hat{i}$ direction respectively.

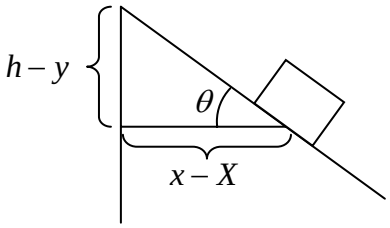
Concept of pseudo force makes Newton's laws & motion valid in non-inertial frames.

1.2.2 Equation of Constrain: limit to motion is known as constrain motion .The relationship between variable which .Actually equation of constraint is used to find independent variable which will describe independent motion of the system. Constraint equations are independent of applied force. It is very important tool to find relationship between acceleration between different particles. which can be better understand by following examples

Example: A block moves on the wedge which in turns moves on a horizontal table. as shown in sketch, if the wedge angle is θ. How the acceleration of wedge and block are related?

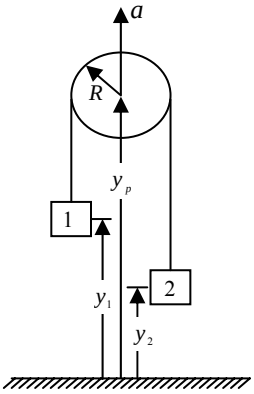


Solution: Let the X be the horizontal coordinate of the end of the wedge and x and y be the horizontal and vertical coordinate of the block as shown in figure. Let h is height of the wedge. So from geometry $(x-X)=(h-y)\cot\theta$ differentiation with respect to t



$$\ddot{x}-\ddot{X}=-\ddot{y}\cot\theta$$

Example: Two masses are connected by a string which passes over a pulley accelerating upward at rate a as shown in figure. If acceleration of the body 1 is a_1 and body 2 is a_2 what is relation between a, a_1 and a_2?

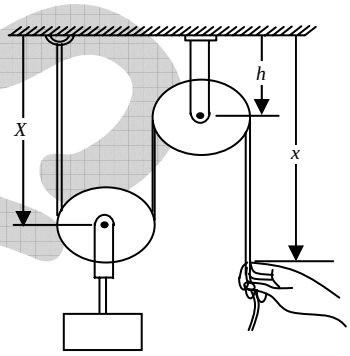


Solution: We know that length of string is fixed so from the figure, $l = \pi R + (y_p - y_1) + (y_p - y_2)$ differentiation with respect to t

$0 = 2\ddot{y}_p - \ddot{y}_1 - \ddot{y}_2$

$\ddot{y}_p = a$, $\ddot{y}_1 = a_1$, $\ddot{y}_2 = a_2$ then $a = \dfrac{a_1 + a_2}{2}$.

Example: the pulley system shown is fig how the acceleration of the rope compare the acceleration of the block. Let us assume acceleration of the block is a and acceleration of end of rope is a_r.



Again the length of rope is fixed let say l. From the fig

$l = X + \pi R + (X - h) + \pi R + (x - h)$.

Where R is radius of pulley .differentiating with respect to

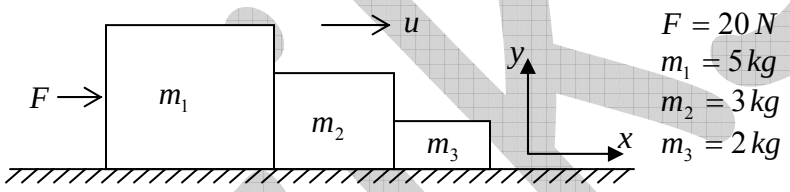
$0 = 2\ddot{X} + \ddot{x} \Rightarrow \ddot{X} = -\dfrac{\ddot{x}}{2}$ if $\ddot{X} = a, \ddot{x} = a_r$ then $a = -\dfrac{a_r}{2}$

1.2.3 Equation of Motion

Finally one should write equation of motion as $\sum F_x = ma_x, \sum F_y = ma_y$ where $\sum F_x$ is summation of external force is external force in x direction and $\sum F_y$ is summation of external force is external force in y direction . a_x, and a_y is acceleration in x and y direction .

Number of equation must be equal to number of unknown variable. so that one can know complete solution .

Example: Force $F = 20N$ act on block $m_1 = 5kg$ such that $m_2 = 3kg$ is in contact with m_1 and $m_3 = 2kg$ as shown in figure.



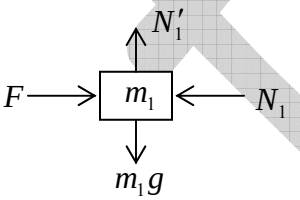
(i) Find acceleration of each block

(ii) Find contact force in each block

Solution: All are moving with same acceleration

$$F = (m_1 + m_2 + m_3)a \Rightarrow a = 2m/\sec^2$$

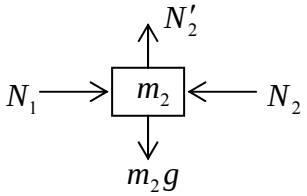
Free body diagram



$F - N_1 = m_1 a$

$20 - 10 = N_1$

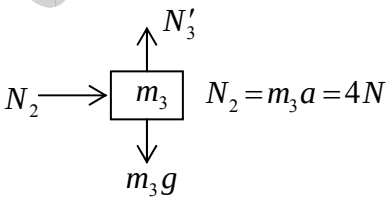
$N_1 = 10N$



$N_1 - N_2 = m_2 a$

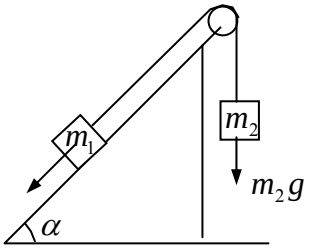
$10 - 4 = N_2$

$N_2 = 6N$



$N_2 = m_3 a = 4N$

Example: The following parameters of the arrangement of are available: the angle α which the inclined plane forms with the horizontal, and the coefficient of friction k between the body m_1 and the inclined plane. The masses of the pulley and the threads, as well as the friction in the pulley, are negligible. Assuming both bodies to be motionless at the initial moment, find the mass ratio m_2 / m_1 at which the body m_2



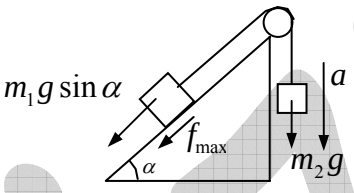
(a) starts coming down;

(b) starts going up;

(c) is at rest.

Solution: (a) for m_2 Starts coming down. For mass m_2 , $m_2 g > T$ and for mass m_1 moving up

$T > m_1 g \sin\alpha + f_{\max}$



therefore $m_2 g > m_1 g \sin\alpha + f_{\max}$

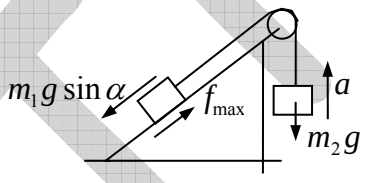
$m_2 g > m_1 g \sin\alpha + k_1 m_1 g \cos\alpha$

$\frac{m_2}{m_1} > \sin\alpha + k\cos\alpha$

for m_2 Starts coming up. For mass m_2 , $m_2 g < T$ and for mass m_1 moving up

$T < m_1 g \sin\alpha - f_{\max}$

(b)



$m_1 g \sin\alpha > m_2 g + k m_1 g \cos\alpha$

$g \sin\alpha > \frac{m_2}{m_1} + k\cos\alpha$

$\frac{m_2}{m_1} < g \sin\alpha - k\cos\alpha$

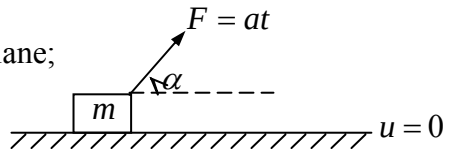
(c) At rest :friction will be static:

$$g \sin\alpha - k\cos\alpha < \frac{m_2}{m_1} < \sin\alpha + k\cos\alpha$$

Example: At the moment $t = 0$ the force $F = at$ is applied to a small body of mass m resting on smooth horizontal plane (a is a constant). The permanent direction of this force forms an angle α with the horizontal Find:

(a) the velocity of the body at the moment of its breaking off the plane;

(b) the distance traversed by the body up to this moment.



Solution:

(a) If one will draw the free body diagram $F\sin\alpha + N = mg$ in y direction

$F\cos\alpha = ma_1$ in x direction .

At time of breaking off the plane vertical component of $\vec{F}$ must be equal to weight mg. Then

$$F\sin\alpha = mg = at\sin\alpha$$

$$t = \frac{mg}{a\sin\alpha}$$

Motion equation of block: a_1 = Acceleration of block

$$F\cos\alpha = m\,a_1$$

$$a_1 = \frac{at\cos\alpha}{m} = \frac{dv}{dt}$$

$$\int_0^V \frac{m\,dv}{a\cos\alpha} = \int_0^{t_1} t\,dt \Rightarrow \frac{mv}{a\cos\alpha} = \frac{t_1^2}{2}$$

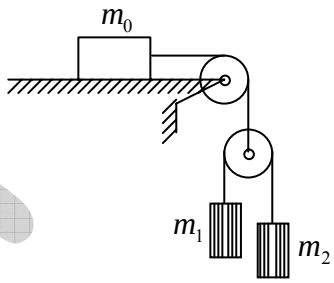
$$\frac{mv}{a\cos\alpha} = \frac{1}{2}\frac{m^2g^2}{a^2\sin^2\alpha} \Rightarrow v = \frac{mg^2\cos\alpha}{2a\sin^2\alpha}$$

(b) $$\int_0^V \frac{m\,dv}{a\cos\alpha} = \int_0^t t\,dt$$

$$\frac{mv}{a\cos\alpha} = \frac{t^2}{2} \Rightarrow v = \frac{a\cos\alpha}{2m}t^2 \Rightarrow \frac{ds}{dt} = \frac{a\cos\alpha\,t^2}{2m}$$

$$\int_0^s ds = \frac{a\cos\alpha}{2m}\int_0^t t^2\,dt \quad \Rightarrow s = \frac{a\cos\alpha}{2}\frac{m}{}\frac{t^2}{\times 3} \Rightarrow s = \frac{m^2g^3\cos\alpha}{6a^2\sin^3\alpha}$$

Example: In the arrangement shown in fig the bodies have masses m_0, m_1, m_2, the friction is absent, the masses of the pulleys and the threads are negligible. Find the acceleration of the body m_1. Look into possible cases.



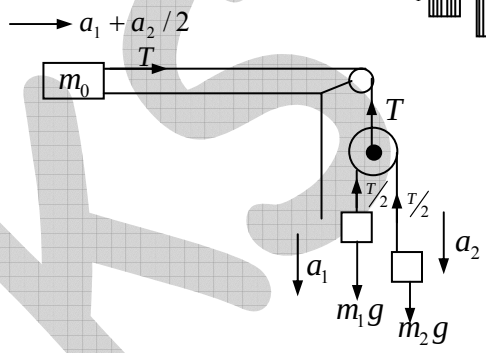
Solution: $T = 2T_1$ and from equation of constrain $a_0 = \dfrac{a_1 + a_2}{2}$

Equation of motion:

$$T = m_0\left(\frac{a_1 + a_2}{2}\right) \qquad \ldots(\text{i})$$

$$m_1 g - T/2 = m_1 a_1 \qquad \ldots(\text{ii})$$

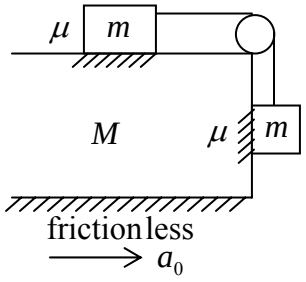
$$m_2 g - T/2 = m_2 a_2 \qquad \ldots(\text{iii})$$



from (i), (ii) and (iii)

$$a_1 = \frac{4m_1m_2 + m_0(m_1 - m_2)g}{4m_1m_2 + m_0(m_1 + m_2)}$$

Example: Coefficient of friction is μ. What will be acceleration of table such that system will be in equilibrium?



Solution: This whole wedge is moving with acceleration a_0.

$$N = mg$$

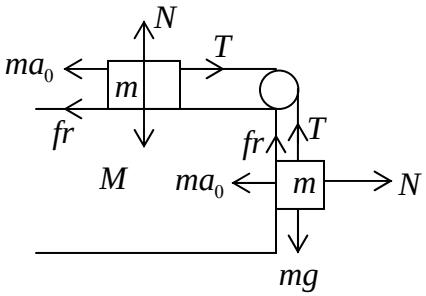
$$ma_0 + fr = T \Rightarrow ma_0 + \mu N = T$$

$$ma_0 + \mu mg = T \Rightarrow m(a_0 + \mu g) = T \qquad (1)$$

$$N - ma_0 = 0 \Rightarrow N = ma_0$$

$$fr + T = mg \Rightarrow T = mg - fr, \quad -\mu ma_0 + mg = T$$

$$\Rightarrow \quad m(-\mu a_0 + g) = T \qquad (2)$$



From (1) and (2) $ma_0 + m\mu g - mg + m\mu a_0 = 0$

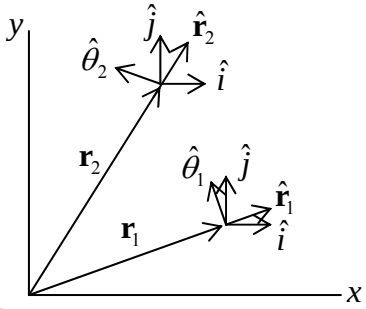
$$-g(1-\mu) + a_0(\mu + 1) = 0 \Rightarrow a_0 = g\frac{(1-\mu)}{(1+\mu)}$$

Head office
fiziks, H.No. 23, G.F, Jia Sarai,
Near IIT, Hauz Khas, New Delhi-16
Phone: 011-26865455/+91-9871145498

Branch office
Anand Institute of Mathematics,
28-B/6, Jia Sarai, Near IIT
Hauz Khas, New Delhi-16

Website: www.physicsbyfiziks.com
Email: fiziks.physics@gmail.com

1.3 The Motion in Two Dimensional in Polar Coordinate



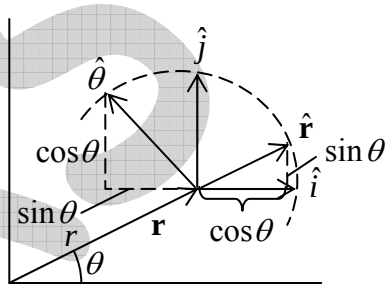
The drawing shows the unit vectors $\hat{i}, \hat{j}$ and $\hat{r}, \hat{\theta}$ at a point in the xy plane. We see that the orthogonality of $\hat{\mathbf{r}}$ and $\hat{\theta}$ plus the fact that they are unit vectors,

$|\hat{\mathbf{r}}| = 1, |\hat{\theta}| = 1$,

$\hat{\mathbf{r}} = \hat{i}\cos\hat{\theta} + \hat{j}\sin\theta$ and $\hat{\theta} = -\hat{i}\sin + \hat{j}\cos\theta$ which is shown

The transformation can be shown by rotational Matrix

$$\begin{bmatrix} \hat{r} \\ \hat{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \hat{i} \\ \hat{j} \end{bmatrix}$$



$$\frac{d\hat{r}}{dt} = -\hat{i}\sin\theta\dot{\theta} + \hat{j}\cos\theta\dot{\theta} \Rightarrow \frac{d\hat{r}}{dt} = \dot{\theta}\hat{\theta}$$

$$\frac{d\hat{\theta}}{dt} = -\hat{i}\cos\theta\dot{\theta} - \hat{j}\sin\theta\dot{\theta} \Rightarrow \frac{d\hat{\theta}}{dt} = -\hat{r}\dot{\theta}$$

1.3.1 The Position Vector in Polar Coordinate:

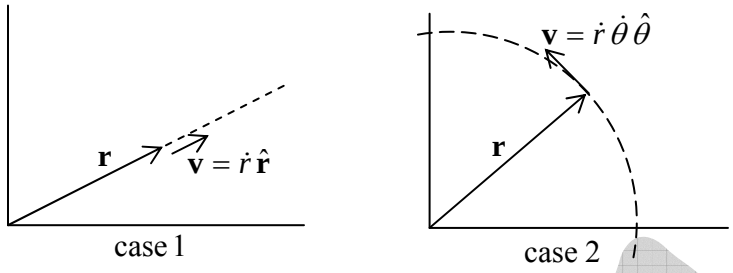
$$\vec{r} = x\hat{i} + y\hat{j}\ \vec{r} = |r|\left[\cos\theta\hat{i} + \sin\theta\hat{j}\right]$$

$$\hat{r} = |r|\cos\theta\hat{i} + |r|\sin\theta\hat{j}$$

$$\vec{r} = |r|\left[\cos\theta\hat{i} + \sin\theta\hat{j}\right] \Rightarrow \vec{r} = |r\ |\ \hat{r},$$

$\mathbf{r} = r\hat{\mathbf{r}}$ is sometimes confusing, because the equation as written seems to make no reference to the angle θ. We know that two parameters needed to specify a position in two dimensional space (in Cartesian coordinates they are x and y), but the equation $\mathbf{r} = r\hat{\mathbf{r}}$ seems to contain only the quantity r. The answer is that $\hat{\mathbf{r}}$ is not a fixed vector and we need to know the value of θ to tell how $\hat{\mathbf{r}}$ is origin. Although θ does not occur explicitly in $r\hat{\mathbf{r}}$, its value must be known to fix the direction of $\hat{\mathbf{r}}$. This would be apparent if we wrote $\mathbf{r} = r\hat{\mathbf{r}}(\theta)$ to emphasize the dependence of $\hat{\mathbf{r}}$ on θ. However, by common conversation $\hat{\mathbf{r}}$ is understood to stand for $\hat{\mathbf{r}}(\theta)$.

1.3.2 Velocity Vector in Polar Coordinate:



$$\vec{v} = \frac{d}{dt}(r\hat{\mathbf{r}}) \Rightarrow \vec{v} = \frac{d(r\hat{r})}{dt} = \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt} = \dot{r}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt} \Rightarrow \vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

where $\dot{r}$ is radial velocity in $\hat{r}$ direction and $r\dot{\theta}$ is tangential velocity in $\hat{\theta}$ direction as shown in figure and the magnitude to velocity vector $|v| = \sqrt{\dot{r}^2 + r^2\dot{\theta}^2}$

1.3.3 Acceleration Vector in Polar Coordinate

$$\frac{d\vec{v}}{dt} = \frac{d\dot{r}}{dt}\hat{r} + \dot{r}\frac{d\hat{r}}{dt} + \frac{dr}{dt}\dot{\theta}\hat{\theta} + r\frac{d\dot{\theta}}{dt}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt}$$

$$\frac{d\vec{v}}{dt} = \ddot{r}\hat{r} + \dot{r}\dot{\theta} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}(-\dot{\theta})\hat{r}$$

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\theta} \Rightarrow \vec{a} = a_r\hat{r} + a_\theta\hat{\theta}$$

$a_r = \ddot{r} - r\dot{\theta}^2$ is radial acceleration and $a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$ is tangential acceleration .

So Newton's law in polar coordinate can be written as

$f_r = ma_r = m(\ddot{r} - r\dot{\theta}^2)$ where f_r is force in radial direction .

$f_\theta = ma_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta})$ where f_θ is force in tangential direction.

1.3.4 Circular Motion

For circular motion $r = r_0$ then $\ddot{r} = 0$ so $f_r = ma_r = -mr_0\dot{\theta}^2$ where f_r is force in radial direction and $f_\theta = ma_\theta = mr_0\ddot{\theta}$ where f_θ is force in tangential direction .

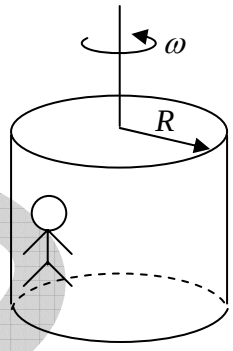
If there is not any force in tangential direction $f_\theta = 0$ is condition then motion is uniform circular motion ie $\dot{\theta} = \omega$ is constant known as angular speed and tangential speed is given by $v = r_0\omega$

For non uniform circular motion radial acceleration is $a_r = -\frac{v^2}{r}$ and tangential acceleration $a_t = \frac{dv}{dt}$ is given by $a_t = \frac{dv}{dt}$

$a = \sqrt{a_r^2 + a_t^2}$

Example: The Spinning Terror

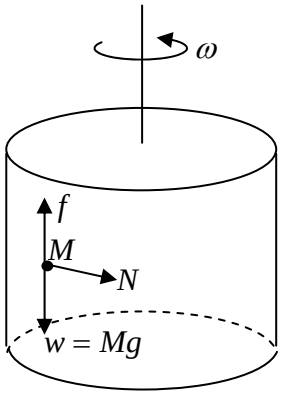
The Spinning Terror is an amusement park ride – a large vertical drum which spins so fast that everyone inside stays pinned against the wall when the floor drops away. What is the minimum steady angular velocity ω which allows the floor to be dropped away safely?



Solution: Suppose that the radius of the drum is R and the mass of the body is M. Let μ be the coefficient of friction between the drum and M. The forces on M are the weight W, the friction force f and the normal force exerted by the wall, N as shown below.

The radial acceleration is $R\omega^2$ toward the axis, and the radial equation of motion is

$$N = M / R\omega^2$$



By the law of static friction,

$$f \le \mu N = \mu MR\omega^2$$

Since we require M to be in vertical equilibrium,

$$f = Mg,$$

and we have

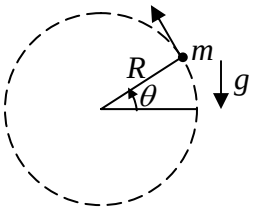
$$Mg \le \mu MR\omega^2$$

or $$\omega^2 \ge \frac{g}{\mu R}$$

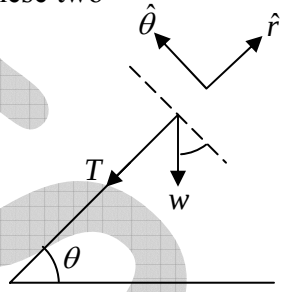
The smallest value of ω that will work is

$$\omega_{\min} = \sqrt{\frac{g}{\mu R}}$$

Example: Mass m is whirled on the end of string R the motion is in a vertical plane in the gravitational field of earth .write down the equation of motion in polar coordinate.



Solution: There are only two forces acted on body one tension T in radial direction and weight mg in vertical direction. Taking component of these two forces in $\hat{r}$ and $\hat{\theta}$ direction.



The Newton's law of motion in radial direction is

$$F_r = ma_r \text{ in } \hat{r} \text{ direction} \Rightarrow -T - mg\sin\theta = m\left(\ddot{r} - r\dot{\theta}^2\right)$$

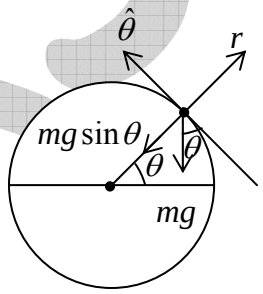
The Newton's law of motion in tangential direction is

$$F_\theta = ma_\theta \Rightarrow -mg\cos\theta = m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)$$

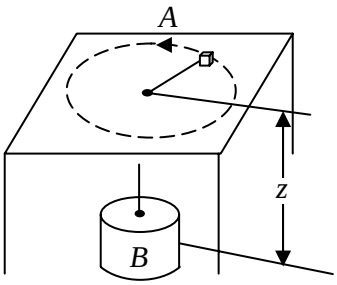
The equation of constrain is $r = R \quad \dot{r} = 0 \quad \ddot{r} = 0$

$$-T - mg\sin\theta = -mR\dot{\theta}^2 \text{ in } \hat{r} \text{ direction}$$

$$-mg\cos\theta = mR\ddot{\theta} \Rightarrow \frac{d^2\theta}{dt^2} = -\frac{g}{R}\cos\theta \text{ in } \hat{\theta} \text{ direction}$$



Example: A horizontal frictionless table has a small hole in the centre of table .Block A of mass m_a on the table is connected by a block B of mass m_b hanging beneath by a string of negligible mass can move under gravity only in vertical direction .which passes through the whole as shown in figure.

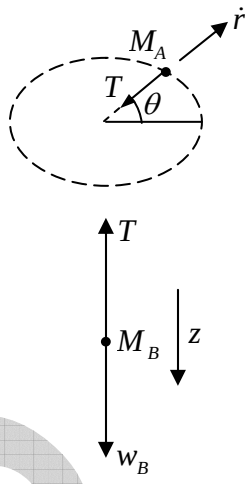


(a) Write down equation of motion in radial, tangential and vertical direction.

(b) What is equation of constrain?

(c) What will be the acceleration of B when A is moving with angular velocity ω at radius r_0 ?

Solution: Suppose of block A rotating in circle with angular velocity ω of radius r_0 what is acceleration of block B assume $\hat{z}$ direction is shown in figure.



The external force in radial direction is tension, not any external force in tangential direction and weight of body m_b and tension in vertical direction.

(a) $m_a(\ddot{r} - r\dot{\theta}^2) = -T$ in $\hat{r}$ direction .....(1)

$m_a(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0$ in $\hat{\theta}$ direction .....(2)

$m\ddot{z} = m_b g - T$ in $\hat{z}$ .....(3)

(b) Since length of the rope is constant then

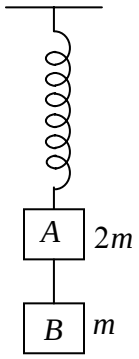
$r + z = l \Rightarrow \ddot{r} = -\ddot{z}$ .....(4)

(c) Put value of $\ddot{r} = -\ddot{z}$, $T = m_a(\ddot{z} + r\dot{\theta}^2)$

$$\ddot{z} = \frac{m_b g - m_a r\dot{\theta}^2}{m_b + m_a} \text{ put } r = r_0, \dot{\theta} = \omega \Rightarrow \ddot{z} = \frac{m_b g - m_a r_0 \omega^2}{m_a + m_b}$$

MCQ (Multiple Choice Questions)

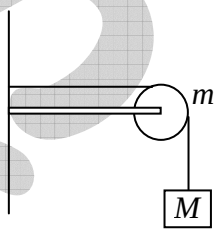
Q1. Two blocks A and B of masses $2m$ and m respectively are connected by a massless and inextensible string. The whole system is suspended by a massless spring. The magnitudes of acceleration of A and B, immediately after the string is cut, are respectively



(a) $g, \frac{g}{2}$ (b) $\frac{g}{2}, g$

(c) g, g (d) $\frac{g}{2}, \frac{g}{2}$

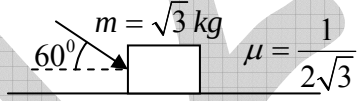
Q2. A string of negligible mass going over a clamped pulley of mass m supports a block of mass M as shown in the figure. The force on the pulley by the clamp is given by



(a) $\sqrt{2}\,Mg$ (b) $\sqrt{2}\,mg$

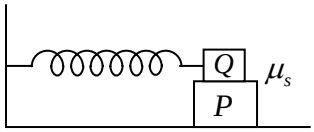
(c) $\sqrt{(M+m)^2+m^2}\,g$ (d) $\sqrt{(M+m)^2+M^2}\,g$

Q3. What is the maximum value of the force F such that the block shown in the arrangement, does not move? Take $g = 10\,m/s^2$.



(a) $20\,N$ (b) $10\,N$ (c) $12\,N$ (d) $15\,N$

Q4. A block P of mass m is placed on a horizontal frictionless plane. A second block of same mass m is placed on it and is connected to a spring of constant k. The two blocks are pulled by distance A. Block Q oscillates without slipping. What is the maximum value of frictional force between the two blocks?



(a) $\frac{kA}{2}$ (b) kA (c) $\mu_s mg$ (d) zero

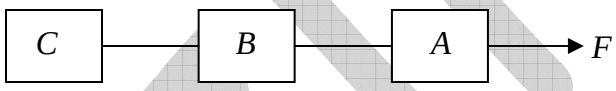
Q5. When forces F_1, F_2 and F_3 are acting on a particle of mass m such that F_2 and F_3 are mutually perpendicular, then the particle remains stationary. If the force F_1 is now removed, then the acceleration of the particle

(a) $\frac{F_1}{m}$ (b) $\frac{F_2 F_3}{mF}$ (c) $\frac{F_2 - F_3}{m}$ (d) $\frac{F_2}{m}$

Q6. A smooth block is released at rest on a 45^0 incline and then slides a distance d. If the time taken to slide on rough incline is n times as large as that to slide on a smooth incline, the coefficient of friction is

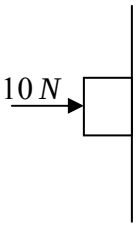
(a) $\frac{n-1}{n}$ (b) $\frac{n-2}{n^2}$ (c) $\frac{n-4}{n^3}$ (d) $\frac{n^2-1}{n^2}$

Q7. Three identical blocks of masses $m = 2\,kg$ each are drawn by a force $10.2\,N$ on a frictionless surface as shown in the figure. What is the tension in the string between the blocks B and C?



(a) 9.2 (b) 8 (c) 3.4 (d) 9.8

Q8. A horizontal force of $10\,N$ is necessary to just hold a block stationary against a wall. The coefficient of friction between the block and the wall is 0.2. The weight of the block is

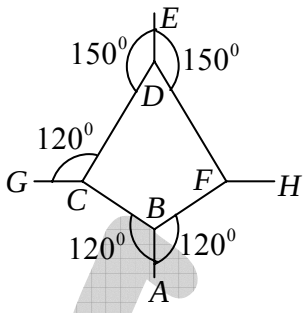


(a) $20\,N$ (b) $50\,N$ (c) $100\,N$ (d) $2\,N$

Q9. The upper half of an inclined plane with inclination θ is perfectly smooth, while the lower half is rough. A body starting from rest at the top will again come to rest at friction for the lower half is given by

(a) $2\tan\theta$ (b) $\tan\theta$ (c) $2\sin\theta$ (d) $2\cos\theta$

Q10. The adjacent figure is a part of a horizontally stretched net. Section AB is stretched with a force of $10\,N$.



The tension in the sections BC and BF are

(a) $10\,N, 11\,N$

(b) $10\,N, 6\,N$

(c) $10\,N, 10\,N$

(d) can not be calculated due to insufficient data.

Q11. A particle is moving in a vertical circle. The tension in the string when passing through two positions at angles 30^0 and 60^0 from vertical (lowest positions) are T_1 and T_2 respectively. Then

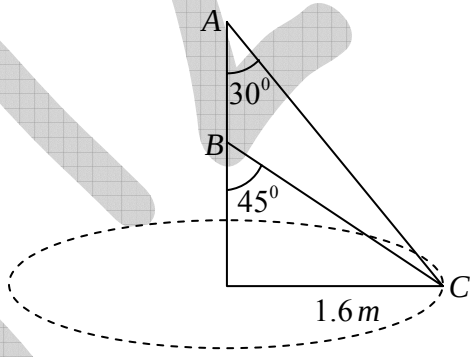
(a) $T_1 = T_2$

(b) $T_2 > T_1$

(c) $T_1 > T_2$

(d) Tension in the string always remains the same.

Q12. Two wires AC and BC are tied to a small sphere C of mass $5\,kg$, which revolved at a constant speed v in the horizontal circle of radius $1.6\,m$. Taking $g = 9.8\,m/s^2$, the minimum value of v is



(a) $3.01\,m/s$ (b) $4.01\,m/s$ (c) $3.2\,m/s$ (d) $3.96\,m/s$

Q13. Three equal weights of $3\,kg$ each are hanging on a string passing over a frictionless pulley as shown in the figure. The tension in the string between the masses II and masses III will be

(a) $5\,N$ (b) $6\,N$ (c) $10\,N$ (d) $20\,N$

Q14. A block of mass m is pulled along a horizontal surface by applying a force at an angle θ with the horizontal. If the block travels with a uniform velocity and has a displaced d and the coefficient of friction is μ, then the work done by the applied force is

(a) $\dfrac{\mu mgd}{\cos\theta+\mu\sin\theta}$ (b) $\dfrac{\mu mgd\cos\theta}{\cos\theta+\mu\sin\theta}$

(c) $\dfrac{\mu mgd\sin\theta}{\cos\theta+\mu\sin\theta}$ (d) $\dfrac{\mu mgd\cos\theta}{\cos\theta-\mu\sin\theta}$

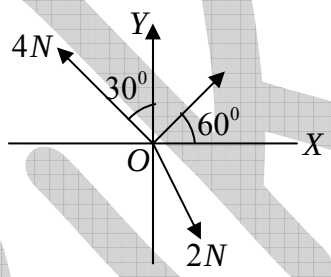
Q15. Three forces acting on a body are shown in the figure. To have the resultant force only along the y - direction, the magnitude of minimum additional force needed along OX is

(a) $0.5\,N$

(b) $1.5\,N$

(c) $\dfrac{\sqrt{3}}{4}N$

(d) $\sqrt{3}\,N$



Q16. An object motion is governed by $v=(\cos\omega t)\hat{i}-\sin(\omega t)\hat{j}$. What is the speed of the object?

(a) ω (b) 2ω (c) 0 (d) 1

Q17. A particle motion in the $x-y$ plane is governed by its position vector $r=a\sin\omega t\hat{i}+b\cos\omega t\,\hat{j}$ where a,b and ω are positive constants and $a>b$. What is the trajectory of the particle?

(a) A parabola (b) A hyperbola

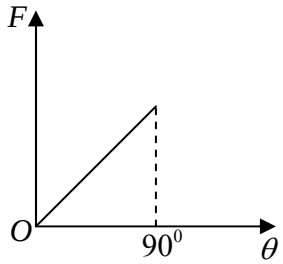
(c) An ellipse (d) A circle

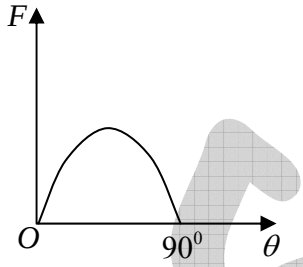
Q18. Consider a particle of mass m, moving on the plane Its position vector is given $\vec{r}=a\cos\omega t\,\hat{i}+b\sin\omega t\,\hat{j}$ where a,b,ω are position constants and $a>b$. What is the direction of force?

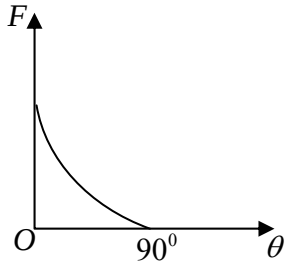
(a) Always towards the origin (b) Always away from the origin

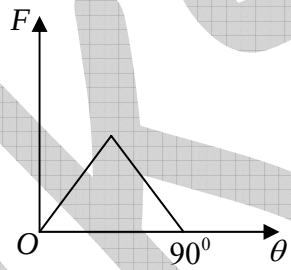
(c) Tangential to its trajectory (d) Unknown

Q19. A block tests on a rough plane, whose inclination θ to the horizontal can be varied. Which of the following graph indicates how the frictional force F between the block and plane varies as θ increased?

(a)


(b)


(c)


(d)


Q20. If a body of mass $45\ kg$ resting o a rough horizontal surface can be just moved by a force of $10\ kg$ weight acting horizontally, then the coefficient of sliding friction is

(a) 4.5 (b) 0.5 (c) 0.45 (d) 0.22

Q21. A block B is resting on a. horizontal plate in the xy plane and the coefficient of friction between the block and the plate is μ. The plate begins to move in the x-direction and its velocity is $u = bt^2$ being time and b being a constant. At what time will the block start sliding on the plate?

(a) $\frac{\mu b}{g}$ (b) $\frac{\mu g b}{2}$ (c) $\frac{\mu g}{b}$ (d) $\frac{\mu g}{2b}$

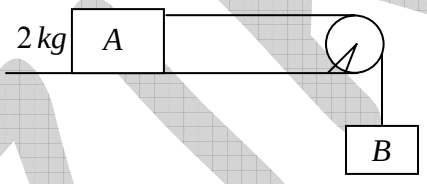
NAT (Numerical Answer Type)

Q22. The force required to accelerate a car of mass $2000\ kg$ from rest to $30\ m/s$ in $12\,s$, if the frictional force between the tyres and the ground is $0.2\ N/kg$ _____ N

Q23. A wooden block having a mass of $1\ kg$ is placed on a table. The block just starts to move, when a force of $10\ N$ is applies at 45^0 to the vertical to pull the block. The coefficient of friction between the table and the block, (taking $g = 10\ m/s^2$) approximate _____.

Q24. The coefficient of static friction between a box and the flat bed of a track is 0.75. the maximum acceleration the truck can have along the level ground if the box is not to slide is_____ m/s^2

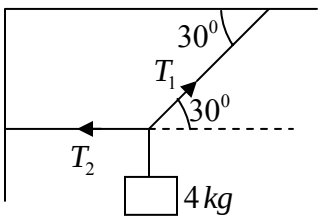
Q25. The coefficient of static friction between block A of mass $2\,kg$ and the table is 0.2. Take $g = 10\,m/s^2$. The maximum mass value of block B so that the two blocks do not move is _____.



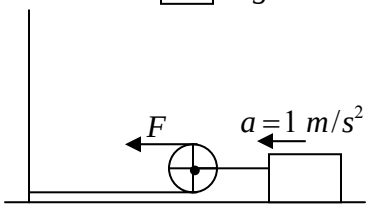
Q26. A block of mass $8\,kg$ is placed on a smooth wedge of inclination 30^0. The whole system is accelerated horizontally so that the block does not slip on the wedge. If we take $g = 10\,m/s^2$, the force exerted by the wedge on the block will be _____ N.

Q27. A person of mass $60\,kg$ is inside a lift of mass $940\,kg$ and presses the button on control panel. The lift starts moving upwards with an acceleration $1.0\,m/s^2$. If $g = 10\,m/s^2$, the tension in the supporting cable is _____ N.

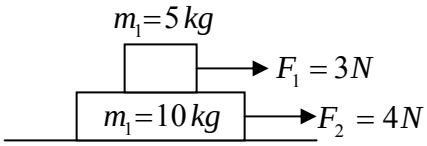
Q28. A block of mass $4\,kg$ hangs as shown in the figure. If we take $g = 9.8\ m/s^2$, value of T_2 is _____.



Q29. A block of mass $200\,kg$ is set into motion on a frictionless horizontal surface with the help of a frictionless pulley and rope system as shown in figure. The horizontal force F that should be applied to produce in the block an acceleration of $1\,m/s^2$ is_____ N.

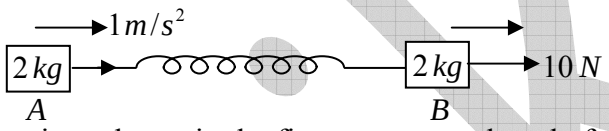


Q30. Two blocks of masses $5\,kg$ and $10\,kg$ are acted on by forces of $3\,N$ and $4\,N$ respectively as shown in the figure.

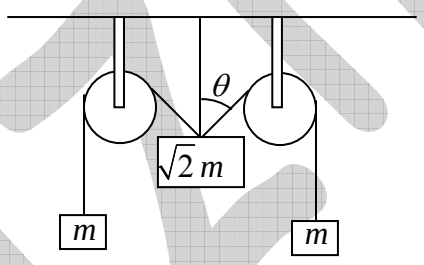


Assuming there is no sliding between the blocks and the ground is smooth, the static friction between the blocks is _____

Q31. Two bodies A and B each of mass $2\,kg$ are connected by a massless spring. A force of $10\,N$ acts on mass B as shown in the figure. At the instant shown body A has acceleration of $1\,m/s^2$, the acceleration of B is _____ m/s^2.



Q32. The pulleys and strings shown in the figure are smooth and of negligible mass. For the system to remain in equilibrium the value of θ is _____ degree.



Q33. A horizontal force of $20\,N$ is applied to a block of mass $4\,kg$ resting on a rough horizontal table. If the block does not move on the table, the frictional force from the table on the block is _____ N. (Take $g=10\,m/s^2$)

Q34. A block slides down an incline at angle 30^0 with an acceleration $\frac{g}{4}$. The kinetic friction coefficient is _____.

MSQ (Multiple Select Questions)

Q35. The force exerted by the floor of an elevator on the foot of a person standing there is more than the weight of the person if the lift is

(a) going up and slowing down (b) going up and speeding down

(c) going down and slowing down (d) going down and speeding up

Q36. If the tension in the cable is supporting an elevator is equal to the weight of the elevator, the elevator may be

(a) going up with increasing speed. (b) going down with increasing speed

(c) going up with uniform speed (d) going down with uniform speed

Q37. A block of mass $20\,kg$ is placed gently on an incline plane having an angle of inclination 37^0. The coefficient of friction between the block and the incline is 0.1. Take $g = 10\,m/s^2$, $\sin 37^0 = \frac{3}{5}$ and $\cos 37^0 = \frac{4}{5}$. Then

(a) The frictional force acting on the block is $16\,N$

(b) The normal force acting on the block is $120\,N$

(c) A force of static friction acts on the block

(d) The acceleration of the block is $5.2\,m/s^2$

Q38. A block of mass $2.5\,kg$ is kept on a rough horizontal surface. It is found that the block does not slide if a horizontal force less than $15\,N$ is applied to it. Also it is found that it takes $5\,s$ to slide through the first $10\,m$, if a horizontal force of $15\,N$ is applied and the block is gently pushed to start the motion. If $g = 10\,m/s^2$, then

(a) the coefficient of static friction is 0.50

(b) the coefficient of static friction is 0.60

(c) the coefficient of kinetic friction is 0.52

(d) the coefficient of kinetic friction is 0.40

Q39. A person applies a constant force $\vec{F}$ on a particle of mass m and finds that the particle moves in a circle of radius r with a uniform speed v as seen from an inertial frame of reference

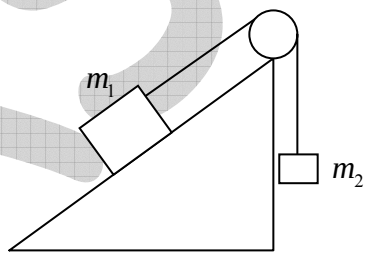
(a) This is not possible

(b) There are other forces on the particle

(c) The resultant of other forces is $\frac{mv^2}{r}$ towards the centre.

(d) The resultant of other forces varies in magnitude as well as in direction.

Q40. Two bodies of masses m_1 and m_2 are connected by a light string going over a smooth light pulley at the end of a frictionless incline as shown in the figure. The whole system is at rest.



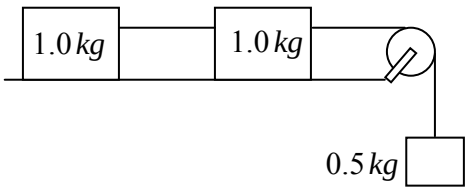
(a) The angle of inline is $\sin^{-1}\left(\frac{m_1}{m_2}\right)$

(b) The angle of inline is $\sin^{-1}\left(\frac{m_2}{m_1}\right)$

(c) The tension in the string is $\sin^{-1}\left(\frac{m_1}{m_2}\right)$

(d) The normal force acting on mass m_1 is $m_1 g\sqrt{1-\left(\frac{m_2}{m_1}\right)^2}$

Q41. A system of three blocks are connected together by means of two inextensible strings. Two blocks lie on a horizontal surface and one of them passes over a smooth pulley as shown in the figure. The coefficient of friction between the blocks and the surface is 0.2. Take $g = 10\,m/s^2$.



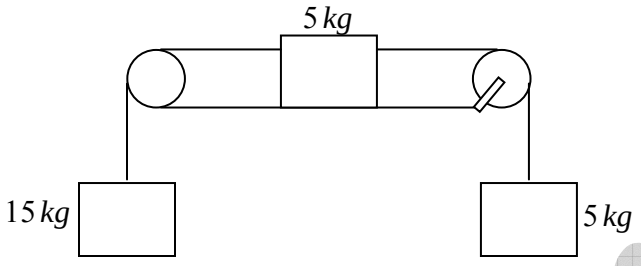
(a) The magnitude of acceleration of the system is $0.4\,m/s^2$

(b) The magnitude of acceleration of the system is $0.8\,m/s^2$

(c) The tension is the string connecting the two $1\,kg$ blocks is $2.4\,N$

(d) The tension in the string attached to $0.5\,kg$ block is $4.8N$

Q42. The friction coefficient between the table and the block shown in the figure is 0.2.



Hence the system can not be in equilibrium. Suppose it moves such that the $15\,kg$ block moves downwards with an acceleration a. In this case, from the free body diagrams.

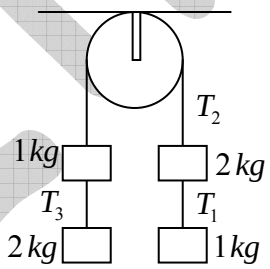
(a) The system remains in equilibrium

(b) The system moves with an acceleration of $\frac{18}{5} m/s^2$

(c) The tension in the left string is $98N$ in the left string.

(d) The tension in the right string is $68\,N$

Q43. In the figure shown all the strings are massless, and friction is absent everywhere. Choose the correct option(s).



(a) $T_1 > T_3$

(b) $T_3 > T_1$

(c) $T_2 > T_1$

(d) $T_2 > T_3$

Q44. A block of mass $10\,kg$ slides down an inclined surface of inclination 30^0. Starting from rest it moves $8\,m$ in the first two seconds. Take $g = 10\,m/s^2$.

(a) The acceleration of the block is $4\,m/s^2$

(b) The coefficient of kinetic friction is 0.1

(c) The kinetic frictional force acting on the block is $50N$

(d) The kinetic frictional force acting on the block is $10N$

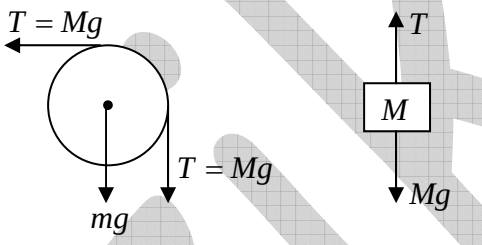
Solutions
MCQ (Multiple Choice Questions)

Ans. 1: (c)

Solution: Immediately after cutting the force on mass m is mg, hence its acceleration is g. Immediately after cutting the spring is unstretched and the force on mass $2m$ is $2mg$, hence its acceleration is g.

Ans. 2: (d)

Solution: For the block of mass M to be in equilibrium, the tension in the string $T = Mg$. From the free body diagram of the block $T = Mg$. From the free body diagram of the pulley



$$F = \sqrt{(Mg + mg)^2 + (Mg)^2} = \sqrt{(M+m)^2 + M^2}\, g$$

Ans. 3: (a)

Solution: For vertical equilibrium of the block,

$$R = mg + F\sin 60^0 = \sqrt{3}g + \frac{\sqrt{3}}{2}F$$

For no motion of the block,

$$f \geq F\cos 60^0 \quad \text{or} \quad \mu R \geq F\cos 60^0$$

$$\text{or} \quad \frac{1}{2\sqrt{3}}\left(\sqrt{3}g + \frac{\sqrt{3}}{2}F\right) \geq \frac{F}{2} \quad \text{or} \quad g \geq \frac{F}{2}$$

$$\text{or} \quad F \leq 2g \quad \text{or} \quad F_{\max} = 20\,N$$

Ans. 4: (a)

Solution: The block Q oscillates but does not slip on P. This indicates acceleration is for both Q and P. A force of friction acts between the two blocks but the horizontal plane is frictionless. The system P - Q oscillates with angular frequency,

$$\omega = \sqrt{\frac{k}{m+m}} = \sqrt{\frac{k}{2m}}$$

Maximum acceleration of the system will be

$$a_{\max} = \omega^2 A = \frac{kA}{2m}$$

This acceleration is provided to the lower block by the force of friction.

$$f_{\max} = ma_{\max} = m\frac{kA}{2m} = \frac{kA}{2}$$

Ans. 5: (a)

Solution: As the particle remains stationary under the action of the three forces, so

$$\vec{F}_1 + \vec{F}_2 + \vec{F}_3 = 0$$

when force $\vec{F}_1$ is removed, the net force left is

$$\vec{F}_2 + \vec{F}_3 = -\vec{F}_1$$

Acceleration $\vec{a} = -\frac{\vec{F}_1}{m}$, hence magnitude of acceleration is $a = \frac{F_1}{m}$.

Ans. 6: (d)

Solution: When there is no friction, the block slides down the inclined plane with acceleration

$$a = g\sin\theta$$

when, there is friction, the downward acceleration of the block is

$$a' = g\left(\sin\theta - \mu\cos\theta\right)$$

As the block slides a distance d in each case, so

$$d = \frac{1}{2}at^2 = \frac{1}{2}a't'^2$$

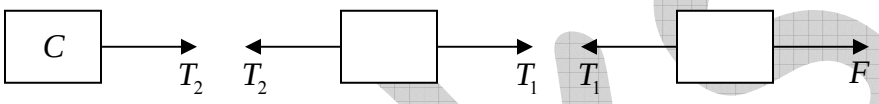
or $\frac{a}{a'}=\frac{t'^2}{t^2}=\frac{(nt)^2}{t^2}=n^2$ or $\frac{g\sin\theta}{g(\sin\theta-\mu\cos\theta)}=n^2$

or $\sin\theta=n^2\sin\theta-\mu n^2\cos\theta$ or $\mu n^2\cos\theta=(n^2-1)\sin\theta$

or $\mu=\left(1-\frac{1}{n^2}\right)\tan\theta=\left(1-\frac{1}{n^2}\right)\tan 45^0 \Rightarrow \mu=1-\frac{1}{n^2}$

Ans. 7: (c)

Solution: Free body diagrams for three blocks



Applying Newton's second law, second law,

$F-T_1=ma$

$T_1-T_2=ma$

$T_2=ma$

Adding the three equations, we get $F=3ma$

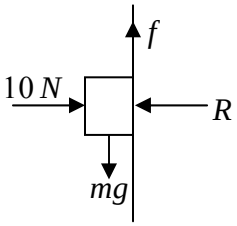
or $a=\frac{F}{3m}$ $\therefore T_2=ma=\frac{F}{3}=\frac{102}{3}=3.4\,N$

Ans. 8: (d)

Solution: For the block to remain stationary,

$R=10\,N,\ f=mg$

but $f=\mu R=0.2\times 10=2\,N$

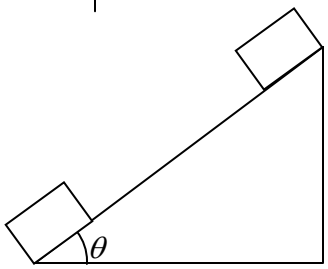


Ans. 9: (a)

Solution: As the initial and final speeds of the block are zero,

Work done by gravity = work done by friction

$mg\sin\theta\times l=f\times\frac{l}{2}$, where l is the length of incline.



or $mg\sin\theta=\frac{\mu mg\cos\theta}{2}$ or $\mu=2\tan\theta$

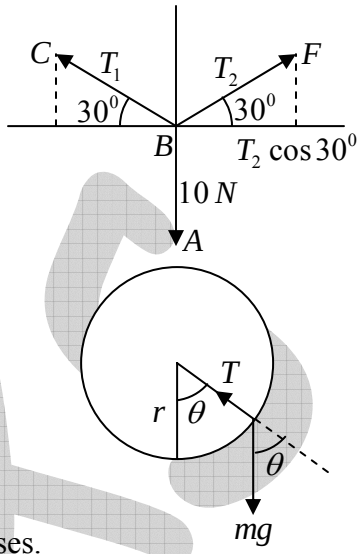
Ans. 10: (c)

Solution: Let T_1 and T_2 be the tension in section BC and BF respectively.

$$T_1\cos 30^0 = T_2\cos 30^0 \Rightarrow \quad T_1 = T_2$$

Also $\quad T_1\sin 30^0 + T_2\sin 30^0 = 10 \Rightarrow 2T_1\sin 30^0 = 10$

$$\Rightarrow \quad 2T_1\times\frac{1}{2} = 10 \quad \Rightarrow T_1 = 10\,N$$



Ans. 11: (c)

Solution: At any angle θ measured from the lowest position

$T - mg\cos\theta = \frac{mv^2}{r}$, where r is the radius of the circle.

or $\quad T = mg\cos\theta + \frac{mv^2}{r}$

As θ increases both $\cos\theta$ and v decrease, hence T decreases.

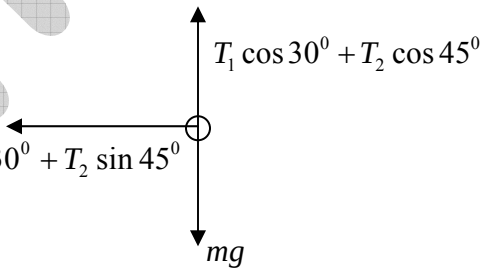
$\therefore \quad T_1 > T_2$

Ans. 12: (d)

Solution: For body diagram fro sphere is shown in the figure.

$$T_1\cos 30^0 + T_2\cos 45^0 = mg \qquad \text{(i)}$$

$$T_1\sin 30^0 + T_2\sin 45^0 = \frac{mv^2}{r} \qquad \text{(ii)}$$



on solving (i) and (ii) $\quad T_1 = \dfrac{mg - \dfrac{mv^2}{r}}{\dfrac{\left(\sqrt{3}-1\right)}{2}}$

$\because \quad T_1 > 0 \Rightarrow mg > \frac{mv^2}{r} \Rightarrow v < \sqrt{rg}$

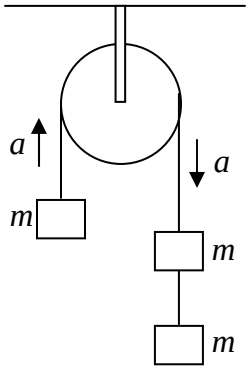
$\therefore \quad v_{max} = \sqrt{rg} = \sqrt{1.6\times 9.8} = 3.96\ m/s$

Ans. 13: (d)

Solution: $T_1 - mg = ma$ (taking the left block as system)

$2mg - T_1 = 2ma$ (Taking the two right blocks as system)

$mg - T_2 = ma$ (Taking the lower block on the right as the system)



$$T_2 = \frac{2}{3}mg = \frac{2}{3}\times 3\times 10 = 20\,N$$

Ans. 14: (b)

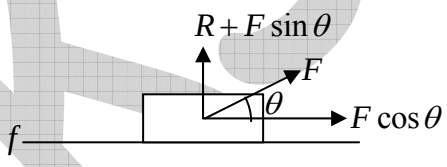
Solution: As the block moves with uniform velocity, the resultant force on it should be zero.

$$R + F\sin\theta = mg \Rightarrow R = mg - F\sin\theta \qquad \therefore \qquad f = \mu R = \mu\left(mg - F\sin\theta\right)$$

Also $F\cos\theta = f$

or $F\cos\theta = \mu\left(mg - F\sin\theta\right)$

or $F\left(\cos\theta + \mu\sin\theta\right) = \mu mg$



$$F = \frac{\mu mg}{\cos\theta + \mu\sin\theta} \qquad \therefore \qquad W = Fd\cos\theta = \frac{\mu mgd\cos\theta}{\cos\theta + \mu\sin\theta}$$

Ans. 15: (a)

Solution: $F + 1\cos 60^0 + 2\cos 60^0 = 4\cos 60^0$

or $F = \left(4 - 1 - 2\right)\cos 60^0 = \frac{1}{2} = 0.5\,N$

Ans. 16: (d)

Solution: $\vec{v}_1 = \cos\omega t\,\hat{i} + \sin\omega t\hat{j}$, $\vec{v}_2 = \cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}$

$$\vec{v}_1 \cdot \vec{v}_2 = \left(\cos\omega t\,\hat{i} - \sin\omega t\,\hat{j}\right)\cdot\left(\cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}\right) = \cos^2\omega t + \sin^2\omega t = 1$$

Ans. 17: (c)

Solution: Equation of trajectory is given by $\vec{r} = x\hat{i} + y\hat{j} = a\sin\omega t\hat{i} + b\cos\omega t\hat{j}$

$x = a\sin\omega t$ and $y = b\cos\omega t$ or $\frac{x}{a} = \sin\omega t$ and $\frac{y}{b} = \cos\omega t$

Squaring and adding $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \sin^2\omega t + \cos^2\omega t = 1$, which is an equation of ellipse.

Ans. 18: (a)

Solution: The position vector of the particle is given as $r = a\cos\omega t\,\hat{i} + b\sin\omega t\,\hat{j}$ where a, b are positive constant with $a > b$

Now $r = a\cos\omega t\,\hat{i} + b\sin\omega t\,\hat{j}$

$$\Rightarrow \quad \frac{dr}{dt} = -a\omega\sin\omega t\hat{i} + b\omega\cos\omega t\,\hat{j} \Rightarrow \frac{d^2r}{dt^2} = -a\omega^2\cos\omega t\hat{i} - b\omega^2\sin\omega t\,\hat{j}$$

$$\Rightarrow \quad \frac{d^2r}{dt^2} = -\omega^2\left[a\cos\omega t\,\hat{i} + b\sin\omega t\,\hat{j}\right] \Rightarrow \frac{d^2r}{dt^2} = \omega^2 r$$

By equation (i)

Force $= m \times$ acceleration

$$\Rightarrow \quad \text{Force} = m\left(\frac{d^2r}{dt^2}\right) \Rightarrow F = m\left(-\omega^2 r\right) \Rightarrow F = -m\omega^2 r \Rightarrow F \propto (-r)$$

Negative sign shows that the direction of force is always towards the origin.

Ans. 19: (a)

Solution: Friction force, $F = \mu mg\sin\theta \Rightarrow F \propto \sin\theta$

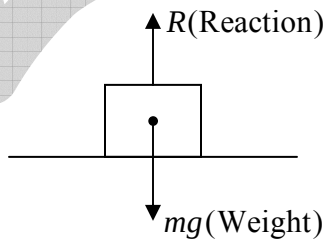
Thus, correct graph is (a)

Ans. 20: (d)

Solution: The force $F = \mu mg$

$$\Rightarrow \quad 10\ g = \mu \times 45\ g$$

$$\Rightarrow \quad \mu = \frac{10}{45} = 0.22$$

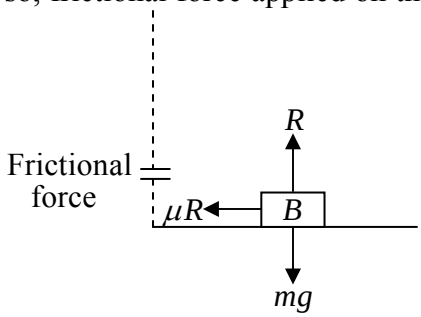


Ans. 21: (d)

Solution: The block B starts to move with velocity $u = bt^2$ so, frictional force applied on this is μR. in $-x$ axis direction as shown.

For starting sliding $F_{\min} \geq$ limiting friction

$$\Rightarrow \quad m\frac{du}{dt} \geq \mu R \Rightarrow m\frac{d}{dt}\left(bt^2\right) \geq \mu mg$$



$\Rightarrow \quad m(2bt) \geq \mu mg \Rightarrow \quad t \geq \dfrac{\mu mg}{2bm}$

$\Rightarrow \quad t \geq \dfrac{\mu g}{2b}$

NAT (Numerical Answer Type)

Ans. 22: $5400\ N$

Solution: $m = 2000\,kg, u = 0,\ v = 30\ m/s,\ t = 12s$

$v = u + at \Rightarrow 30 = 0 + a \times 12 \Rightarrow a = 2.5\ m/s^2$

The force required for acceleration $= 2.5 \times 2000 = 5000\ N$

Friction force $= \mu mg = 0.2 \times 2000 = 400\ N$

Total force $= 5000 + 400 = 5400\ N$

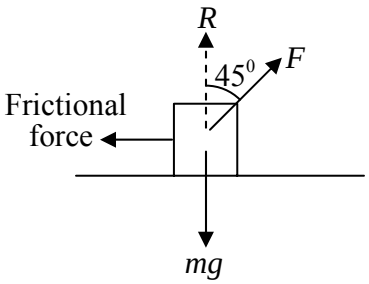
Ans. 23: 0.7

Solution: Resolving the farces and l g their horizontal and vertical component

$mg = R + F\cos\theta$ (i)

Frictional force $= F\sin\theta$ (ii)

By equations (ii), we get

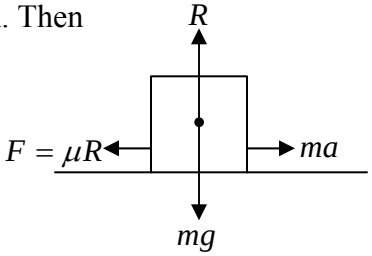


$\Rightarrow \quad \mu mg = F\sin 45^0$

$\Rightarrow \quad \mu = \dfrac{F\sin 45^0}{mg} = \dfrac{10\left(\frac{1}{\sqrt{2}}\right)}{1\times 10} = \dfrac{1}{\sqrt{2}} = 0.707$

Ans. 24: $7.35 m/s^2$

Solution: Writing the equation of motion. If a is acceleration. Then



$ma = F \Rightarrow ma = \mu R \Rightarrow R = mg$

So, $ma = \mu mg \Rightarrow a = \dfrac{m\mu g}{m} \Rightarrow \quad a = \mu g$

Here, $\mu = 0.75, g = 9.8\,m/s^2$

So, acceleration $a = 0.75 \times 9.8 = 7.35\ m/s^2$

Ans. 25: $0.4\,kg$

Solution: For the equilibrium of block A, the force of friction must balance the tension in the string

$\therefore \quad \mu m_A g = T \qquad$ (i)

For the equilibrium of block B, the tension in the string must be equal to its weight

$\therefore \quad m_B g = T$

Hence $m_B g = \mu m_A g$

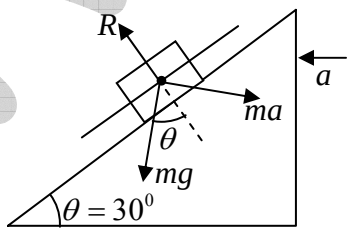
or $m_B = \mu m_A = 0.2 \times 2 = 0.4\,kg$

Ans. 26: $\dfrac{160}{\sqrt{3}}$ or 92.48

Solution: Let the whole system is accelerated horizontally with an acceleration a.

For the block $ma\cos\theta = mg\sin\theta$

or $a = g\tan\theta$



In the direction normal to the plane

$$R = mg\cos\theta + ma\sin\theta = mg\cos\theta + m\cdot\frac{g\sin\theta}{\cos\theta}\cdot\sin\theta$$

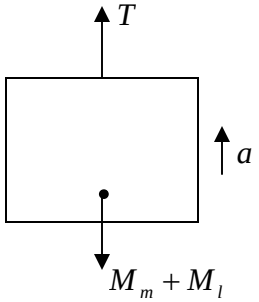
$$= \frac{mg\left(\cos^2\theta + \sin^2\theta\right)}{\cos\theta} = \frac{mg}{\cos\theta}$$

$\therefore \quad R = \dfrac{8\times 10}{\cos 30^0} = \dfrac{8\times 10\times 2}{\sqrt{3}}$ or $\quad R = \dfrac{160}{\sqrt{3}}N = 92.48\,N$

Ans. 27: 11000

Solution: From the free body diagram,

$$T - \left(M_m + M_l\right)g = \left(M_m + M_l\right)a$$

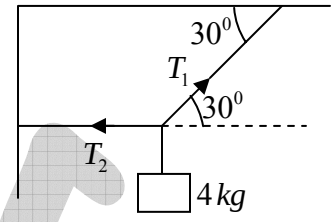


where M_m and M_l are the masses of the man and the lift respectively.

$$T = \left(M_m + M_l\right)\left(g + a\right) = \left(940 + 60\right)\left(10 + 1\right) = 11000\,N$$

Ans. 28: 67.8

Solution: Resolve the tension T_1 along horizontal and vertical directions and as the body is in equilibrium



$$T_1 \sin 30^0 = 4 \times 9.8$$

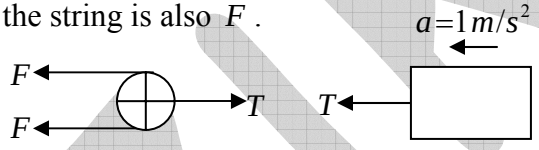
$$T_1 \cos 30^0 = T_2$$

so $$T_1 = \frac{4 \times 9.8}{\sin 30^0} = 78.4 \ N$$

$$T_2 = T_1 \cos 30^0 = \frac{78.4 \times \sqrt{3}}{2} = 39.2\sqrt{3} = 67.8 \ N$$

Ans. 29: 100

Solution: As shown in the force body diagram, when force F is applied at the end of the string, the tension in the lower part of the string is also F.



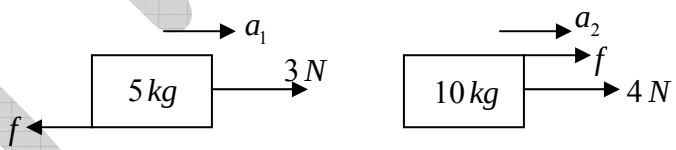
If T is the tension in the string connecting the pulley and the block, then

$$T = 2F \quad \text{or} \quad T = ma = (200)(1) = 200 \ N$$

$$\therefore \quad 2F = 200 \ N \quad \text{or} \quad F = 100 \ N$$

Ans. 30: 0.67

Solution: The free body diagram of the blocks is



From Newton's second law,

$$3 - f = 5a_1 \qquad \text{(i)}$$

$$4 + f = 10a_2 \qquad \text{(ii)}$$

Since the friction is static $a_1 = a_2$

Hence using (i) and (ii)

$$\frac{3-f}{5}=\frac{4+f}{10}$$

this gives $f=\frac{2}{3}=0.67$

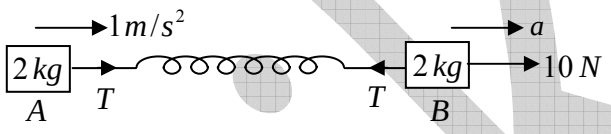
Ans. 31: 4

Solution: Let T be the tension in the string. Then the equation of motion for A is

$$T=2.1=2N$$

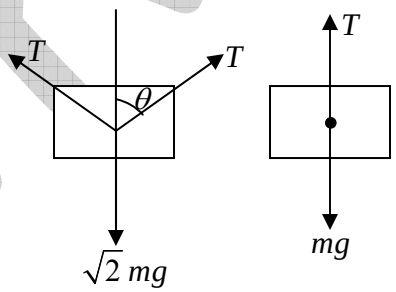
Let a be the acceleration of B. Then

$$2a=10-T\Rightarrow a=4\,m/s^2$$



Ans. 32: 45

Solution: The free body diagrams of masses $\sqrt{2}\,m$ are shown



For the equilibrium of mass m

$$mg=T$$

For the equilibrium of mass $\sqrt{2}\,m$

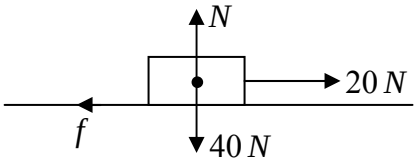
or $$\sqrt{2}\,mg=2T\cos\theta$$

or $$\sqrt{2}\times T=2T\cos\theta$$

$\therefore$ $$\theta=45^0$$

Ans. 33: 20

Solution: The free body diagram is shown in the figure.



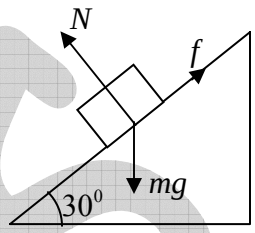
As the block is at rest, these forces should add up to zero. Taking horizontal and vertical components, $f=20\,N$

Ans. 34: $\dfrac{1}{2\sqrt{3}}$

Solution: Free body diagram is shown below

Taking components parallel to the incline and writing Newton's second law

$$mg\sin 30^0 - f = \frac{mg}{4} \Rightarrow f = \frac{mg}{4}$$



There is no acceleration perpendicular to the incline. Hence

$$N = mg\cos 30^0 = mg\frac{\sqrt{3}}{2}$$

as the block is slipping on the incline, friction is

$$f = \mu_k N$$

so $\mu_k = \dfrac{f}{N} = \dfrac{mg}{4mg\cdot\dfrac{\sqrt{3}}{2}}$ or $\mu_k = \dfrac{1}{2\sqrt{3}}$

MSQ (Multiple Select Questions)

Ans. 35: (b) and (c)

Solution: In case (a) acceleration is downwards.

In case (b) acceleration is upwards

In case (c) acceleration is upwards

In case (d) acceleration is downwards

$N = m(g+a)$, hence the force exerted by the floor is more than the true weight.

Ans. 36: (c) and (d)

Solution: We have $T - mg = ma$ or $T = m(g+a)$ for upwards acceleration

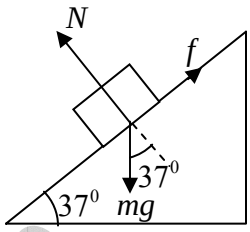
For downwards acceleration $mg - T = ma$ or $T = m(g-a)$

For uniform speed up or down

$T = mg$, since acceleration of the elevator is zero.

Ans. 37: (a), (b) and (d)

Solution: The normal force acting on the block is



$$N = mg\cos 31^0 = 20\times 10\times \frac{4}{5} = 160\,N$$

The maximum value of frictional force acting on the block is

$$f_{\max} = \mu N = 0.1\times 160 = 16\,N$$

The component of weight pulling the block down the incline

$$= mg\sin 37^0 = 20\times 10\times \frac{3}{5} = 120N$$

Since this component is greater than the maximum value of frictional force, the block slides down the incline and the frictional force is at its maximum value.

Hence $f = f_{\max} = 16\,N$

From Newton's second law, $mg\sin 37^0 - f = ma$

or $\quad 20\times 10\times \frac{3}{5} - 16 = 20\times a \Rightarrow 120-16 = 20a \;\Rightarrow a = 5.2\,m/s^2$

Ans. 38: (b) and (c)

Solution: Free body diagram of block is shown

The maximum value of static friction force is $15N$ since the block does not slide up to this force.

Hence $\mu_s N = 15$

or $\quad \mu_s = \frac{15}{N} = \frac{15}{Mg} = \frac{15}{2.5\times 10}$

$\therefore \quad \mu_s = 0.60$

when the block is gently pushed to start the motion, kinetic friction acts between the block and the surface. Since the block takes 5 seconds to slide through the first $10\,m$, the acceleration a given by

$$10\,m = \frac{1}{2}\times a\times (5)^2 \qquad \text{or} \qquad a = 0.8\,m/s^2$$

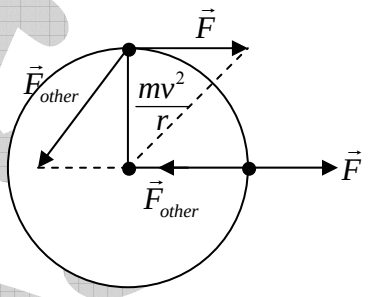
Applying Newton's second law

$$F = -\mu_k Mg = Ma \qquad \text{or} \qquad \mu_k = \frac{F - Ma}{Mg}$$

$$\text{or} \qquad \mu_k = \frac{15N - (2.5)(0.8)}{(2.5)(10)} \qquad \text{or} \qquad \mu_k = 0.52$$

Ans. 39: (b) and (d)

Solution: If the net force acting on the particle is $\frac{mv^2}{r}$ towards the centre the particle moves in a circle of radius r with uniform speed. A single constant force can not always point towards the centre. Hence there are other forces acting on the particle such that the resultant of $\vec{F}$ and there other forces points towards the centre.



Take two positions of the circle. From the figure we see that $\vec{F}_{other}$ must vary in magnitude as well as in direction.

Ans. 40: (b) and (d)

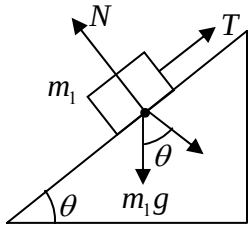
Solution: Free body diagrams of two masses as shown

Let the angle of incline be θ. From the free body diagrams

For mass m_1

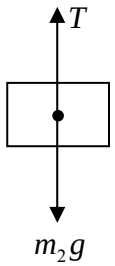
$$N = m_1 g\cos\theta \qquad \text{(i)}$$

and $$T = m_1 g\sin\theta \qquad \text{(ii)}$$



For mass m_2

$$T = m_2 g \qquad \text{(iii)}$$



From (ii) and (iii) $\sin\theta = \frac{m_2}{m_1}$ or $\theta = \sin^{-1}\frac{m_2}{m_1}$

From equation (ii) $T = m_1 g \sin^{-1}\left(\frac{m_2}{m_1}\right)$

From equation (i) $N = m_1 g\sqrt{1-\left(\frac{m_2}{m_1}\right)^2}$

Ans. 41: (a), (b) and (d)

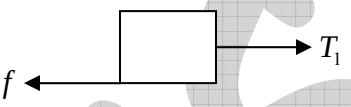
Solution: Let a be the magnitude of the acceleration of the system.

Taking the left $1\,kg$ block as if system

$$T_1 - f = 1.0\times a$$

or $T_1 - 0.2\times 1.0\times 10 = a$

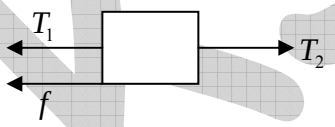
or $T_1 - 2 = a$ (i)



Taking right $10\,kg$ block as if system

$$T_2 - T_1 - f = 1.0\times a$$

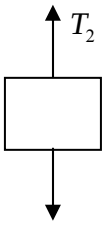
or $T_2 - T_1 - 0.2\times 1.0\times 10 = a$ (ii)



Taking $0.5\,kg$ block as if system

$$0.5\times 10 - T_2 = 0.5\,a$$

or $5 - T_2 = 0.5\,a$ (iii)



putting the values of T_1 from (i) and T_2 from (iii) in equation (ii)

$$(5-0.5a)-(2+a)-2 = a$$

or $2.5a = 1$ or $a = 0.4\,m/s^2$

From equation (i) $T_1 = 2 + a = 2.4\,m/s^2$

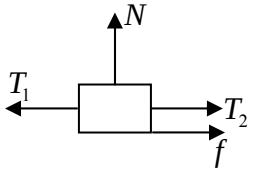
From equation (iii) $T_2 = 5 - 0.5\times 0.4 = 4.8\,m/s^2$.

Ans. 42: (b), (c) and (d)

Solution: Suppose the system is in equilibrium then the regained value of frictional force on the $5\,kg$ block is

$$T_2 + f - T_1 = 0$$

or $f = T_1 - T_2$



but in the case

$T_1 = 15\times 10 = 150N$ and

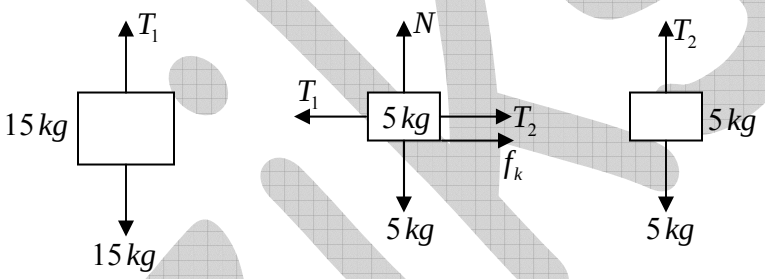
$T_2 = 5\times 10 = 50N$

hence $f = 150 - 50 = 100N$

But in the maximum possible frictional force acting on the middle $5.0\,kg$ mass is

$f_{mass} = 0.2\times 5\times 10 = 10N$

hence the system can not be in equilibrium. Suppose it moves such that the $15\,kg$ block moves downwards with an acceleration a. In this case, from the free body diagrams



$$15\times 10 - T_1 = 15a$$

or $$150 - T_1 = 15a \qquad \text{(i)}$$

$$T_1 - T_2 - f_k = 5a$$

or $$T_1 - T_2 - 0.2\times 5\times 10 = 5a$$

or $$T_1 - T_2 - 10 = 5a \qquad \text{(ii)}$$

$$T_2 - 5\times 10 = 5a$$

or $$T_2 - 50 = 5a \qquad \text{(iii)}$$

Putting T_1 and T_2 from (i) and (iii) respectively in (ii):

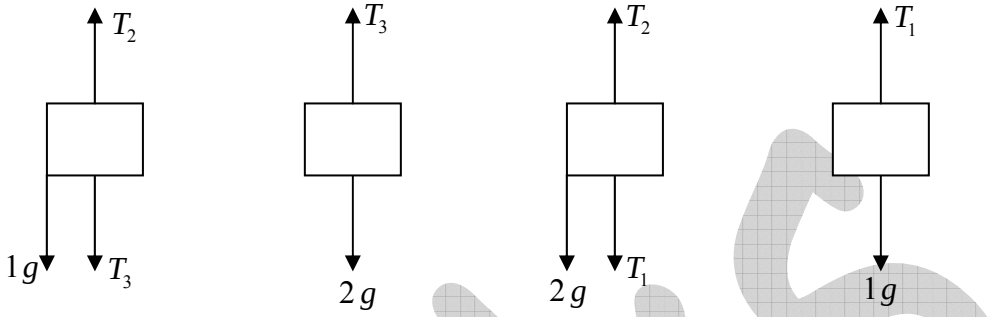
$$(150 - 15a) - (59 - 50) - 10 = 5a \Rightarrow 25a = 90 \text{ or } a = \frac{18}{5} m/s^2$$

$\therefore$ $$T_2 = 50 + 59 = 50 + 5\times\frac{18}{5} = 68N$$

$$T_1 = 150 - 15a = 150 - 15\times\frac{18}{5} = 96N$$

Ans. 43: (b), (c) and (d)

Solution: Let the system moves clockwise with acceleration a. Hence free body diagrams of different blocks.



Left $1kg$ mass Left $2kg$ mass Right $2kg$ mass Right $1kg$ mass

Hence

$$T_2 - 10 - T_3 = a \qquad \text{(i)}$$

$$T_3 - 20 = 2a \qquad \text{(ii)}$$

$$T_1 + 20 - T_2 = 2a \qquad \text{(iii)}$$

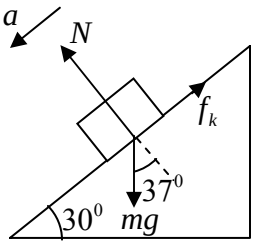
$$10 - T_1 = a \qquad \text{(iv)}$$

Solving these equations gives $a = 0$.

Hence, $T_1 = 10N, T_2 = 30N, T_3 = 20N$

Ans. 44: (a), (b) and (d)

Solution: The free body diagram is shown.



From the relation

$$s = ut + \frac{1}{2}at^2 \Rightarrow 8 = 0 + \frac{1}{2} \times a \times 4 \Rightarrow a = 4\,m/s^2$$

From Newton's second law

$$mg\sin 30^0 - f_k = ma$$

or $\quad f_k = \dfrac{mg}{2} - ma \quad$ or $\quad f_k = \dfrac{10 \times 10}{2} - 10 \times 4 = 10N$

$\therefore$ From the relation $\quad f_k = \mu_k mg \quad$ or $\quad \mu_k = \dfrac{f_k}{mg} = \dfrac{10}{10 \times 10} = 0.1$

2. Energy and Momentum

Conservation of energy:- The energy can neither will created nor destroy it can be transformer from one form to another form.

2.1 Different Form of Energy used in Mechanics

2.1.1 Kinetic energy: The energy which is responsible for motion of the particle. If particle of mass m moving with velocity $\vec{v}$ then kinetic energy is given by $K=\frac{1}{2}m(\vec{v}.\vec{v})$.

Kinetic energy is Cartesian coordinate $K=\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)$

Kinetic energy in cylindrical coordinate $K=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2+\dot{z}^2)$

Kinetic energy in spherical coordinate $K=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2+r^2\sin^2\theta\dot{\phi}^2)$

2.1.2 Potential energy: U – the energy which is required to perform the work is known as potential energy U. Hence force is defined as $F=-\frac{\partial U}{\partial r}$ then potential energy $U_b-U_a=-\int_a^b \vec{F}.d\vec{r}$

for the conservative force one can say potential energy is negative integral of the force .one can say in another way change in potential energy with respect to position is cause of force F.

There is different type of potential energy for example electrostatic potential energy, gravitation potential energy , stored energy in spring , mass energy in relativistic mechanics .

Total energy E is sum of kinetic energy K and potential energy U. So total energy $E=K+U$

Energy diagram ……

2.2 The work-energy Theorem

The work-energy Theorem is given by change in kinetic energy is equal to work done by the system.

The quantity $\frac{1}{2}mv^2$ is called the kinetic energy K, and the change in kinetic energy can be written as K_b-K_a. The integral $\int_{r_a}^{r_b}\vec{F}\cdot d\vec{r}$ is called the work W_{ba} done by the force F on the particle.

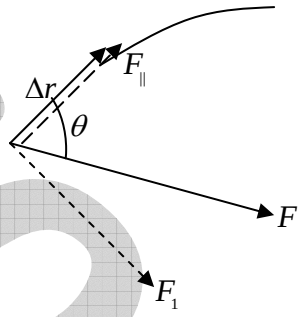
From work energy theorem $W_{ba} = K_b - K_a$.

This result is the general statement of the work-energy theorem which we met in restricted form in our discussion of one dimensional motion.

The work ΔW done by a force F in a small displacement Δr is

$$\Delta W = F \cdot \Delta r = F \cos\theta \Delta r = F_{\parallel} \Delta r ,$$

where $F_{\parallel} = F\cos\theta$ is the component of F along the direction of Δr. The component of F perpendicular to Δr does no work. For a finite displacement from r_a to r_b, the work on the particle, $\int_a^b F \cdot dr$, is the sum of the contribution $\Delta W = F_{\parallel} \Delta r$ from each segment of the path, in the limit where the size of each segment approaches zero.



In the work-energy theorem, $W_{ba} = K_b - K_a$, W_{ba} is the work done on the particle by the total force F. If F is the sum of several forces $F = \Sigma F_i$, we can write

$$W_{ba} = \sum_i \left(W_i\right)_{ba} = K_b - K_a ,$$

where $\left(W_i\right)_{ba} = \int_{r_a}^{r_b} F_i \cdot dr$ is the work done by the i th force F_i.

Work energy theorem in one dimension

we demonstrated the formal procedure for integrating Newton's second law with respect to position. The result was

$$\frac{1}{2}mv_b^2 - \frac{1}{2}mv_a^2 = \int_{x_a}^{x_b} F\left(x\right)dx ,$$

which we now wish to interpret in physical terms.

The quantity $\frac{1}{2}mv^2$ is called the kinetic energy K and the left hand side can be written $K_b - K_a$.

The integral $\int_{x_a}^{x_b} F\left(x\right)dx$ is called the work $W_{ba} = K_b - K_a$.

This result is known as the work-energy theorem or, more precisely, the work-energy theorem in one dimension. (We shall shortly see a more general statement). The unit of work and energy in the SI system is the Joule (J)

$1J = 1kgm^2/s^2$. The unit of work and energy in the cgs system is the erg: $1erg = 1gmcm^2/s^2 = 10^{-7}J$

The unit work in the English system is the foot-pound: $1\ ft \cdot lb \approx 1.336\ J$.

Conservative forces **conservative forces** if $\vec{F}$ is force and it is given that $\vec{\nabla} \times \vec{F} = 0$ then force is said to be non conservative .the work done due to non conservative forces is path **independent**.

2.3 Work Done by a Uniform Force

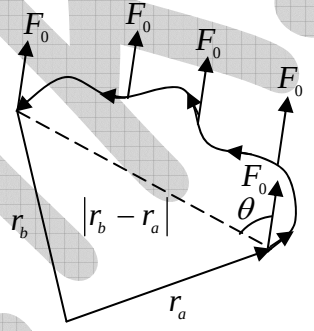
The case of a uniform force is particularly simple. Here is how to find the work done by a force. $F = F_0\hat{n}$ where F_0 is a constant and $\hat{n}$ is a unit vector in some direction, as the particle moves from r_a to r_b along some arbitrary path. All the steps are put in to make the procedure clear, but with any practice this problem can be solved by inspection.

$$W_{BA} = \oint_{r_a}^{r_b} F \cdot dr = \oint_{r_a}^{r_b} F_0\hat{n} \cdot dr = F_0\hat{n} \cdot \oint_{r_a}^{r_b} dr$$

$$= F_0\hat{n} \cdot \left(\hat{i}\int_{x_a,y_a,z_a}^{x_b,y_b,z_b} dx + \hat{j}\int_{x_a,y_a,z_a}^{x_b,y_b,z_b} dy + \hat{k}\int_{x_a,y_a,z_a}^{x_b,y_b,z_b} dz \right)$$

$$= F_0\hat{n} \cdot \left[\hat{i}\left(x_b - x_a\right) + \hat{j}\left(y_b - y_a\right) + \hat{k}\left(z_b - z_a\right) \right]$$

$$= F_0\hat{n} \cdot \left(r_b - r_a\right) = F_0 \cos\theta \left| r_b - r_a \right|$$



For a constant force the work depends only on the net displacement. $r_b - r_a$, not on the path followed. This is not generally the case, but it holds true for an important group of forces,

2.4 Non conservative forces

If $\vec{F}$ is force and it is given that $\vec{\nabla} \times \vec{F} \neq 0$ then force is said to be non conservative .the work done due to non conservative forces is path dependent .

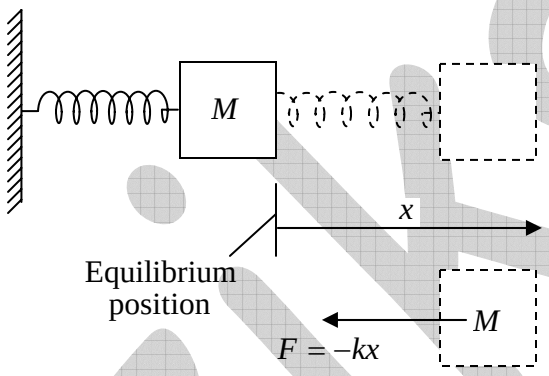
For non conservative force $\vec{F}$ work energy theorem is given by $E_b - E_a = \int_a^b \vec{F}.d\vec{r}$

Where E_b and E_a is total energy at point b and a respectively.

Example: Consider a mass M attached to a spring with spring constant k. Using the coordinate x measured from the equilibrium point x_0 solve the Equation of motion for Simple Harmonic Motion with help of work energy theorem.

Solution: Consider a mass M attached to a spring. Using the coordinate x measured from the equilibrium point, the spring force is $F = -kx$.

$$\frac{1}{2}Mv^2 - \frac{1}{2}Mv_0^2 = -k\int_{x_0}^{x} x\,dx = -\frac{1}{2}kx^2 + \frac{1}{2}kx_0^2$$



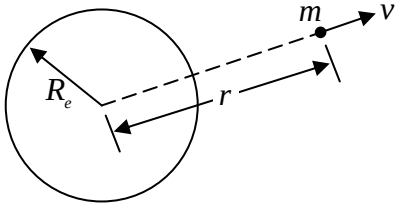
In order to find x and v, we must know their values at some time t_0. Let us consider the case where at $t = 0$ the mass is released from rest, $v_0 = 0$, at a distance x_0 from the origin. Then

$$v^2 = -\frac{k}{M}x^2 + \frac{k}{M}x_0^2 \text{ and } \frac{dx}{dt} = \sqrt{\frac{k}{M}}\sqrt{x_0^2 - x^2}$$

Separating the variables gives. $\int_{x_0}^{x} \frac{dx}{\sqrt{x_0^2 - x^2}} = \sqrt{\frac{k}{M}}\int_0^t dt = \sqrt{\frac{k}{M}}\,t \Rightarrow \sin^{-1}\frac{x}{x_0} = \sqrt{\frac{k}{M}}\,t$

Example A mass m is shot vertically upward from the surface of the earth with initial speed v_0. Assuming that the only force is gravity, find its maximum altitude and the minimum value of v_0 for the mass to escape the earth completely.

Solution: The force on m is $F = -\frac{GM_e m}{r^2}$.



The problem is one dimensional in the variable r and it is simple to find the kinetic energy at distance r by the work-energy theorem.

Let the particle start at $r = R,$ with initial velocity v_0.

$$K(r) - K(R_e) = \int_{R_e}^{r} F(r)\,dr = -GM_e m \int_{R_e}^{r} \frac{dr}{r^2}$$

or $$\frac{1}{2}mv(r)^2 - \frac{1}{2}mv_0^2 = GM_m\left(\frac{1}{r} - \frac{1}{R_e}\right)$$

We can immediately find the maximum height of m. At the highest point, $v(r) = 0$ and we have

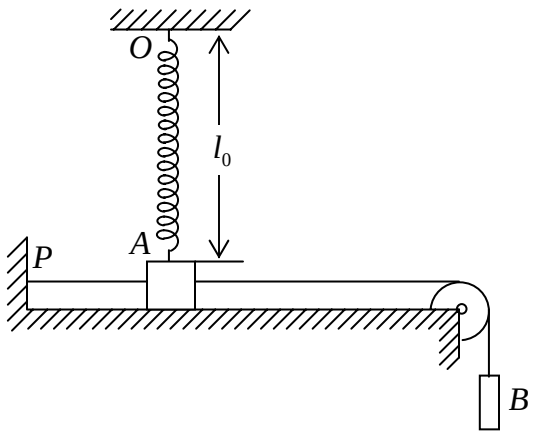
$$v_0^2 = 2GM_e\left(\frac{1}{R_e} - \frac{1}{r_{\max}}\right).$$

It is a good idea to introduce known familiar constants whenever possible. For example, since $g = \frac{GM_e}{R_e^2}$, we can write

$$v_0^2 = 2gR_e^2\left(\frac{1}{R_e} - \frac{1}{r_{\max}}\right) = 2gR_e\left(1 - \frac{R_e}{r_{\max}}\right) \text{ or } r_{\max} = \frac{R_e}{1 - \frac{v_0^2}{2gR_e}}.$$

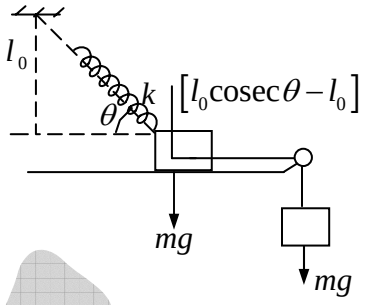
The escape velocity from the earth is the initial velocity needed to move $r_{\max}$ to infinity. The escape velocity is therefore $v_{escape} = \sqrt{2gR_e} = \sqrt{2 \times 9.8 \times 6.4 \times 10^6} = 1.1 \times 10^4\, m/s$

Example: A small bar A resting on a smooth horizontal plane is attached by threads to a point P and by means of a weightless pulley, to a weight B possessing the same mass as the bar itself. Besides the bar is also attached to a point O by means of a light non-deformed spring of length l_0 cm and stiffness $x = 5\, mg/l_0$, where m is the mass of the bar. The thread PA having been burned, the bar starts moving. Find its velocity at the moment when it is breaking off the plane.



Solution: When bay A breaking off the plane then normal reaction will be zero, then

Force equation:



$$kl_0[\cos ec\theta - 1]\sin\theta = mg$$

$$kl_0[1-\sin\theta] = mg$$

$$\sin\theta = \frac{kl_0 - mg}{k\ell_0} = 1 - \frac{mg}{kl_0} \qquad \ldots(\text{i})$$

Energy equation: from work energy theorem

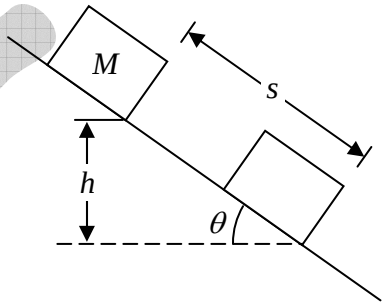
Loss in P.E (mg) = Gain in K.E. of both block + spring energy.

$$mg\, l_0 \cot\theta = \left(\frac{1}{2}mv^2\right) + \left(\frac{1}{2}mv^2\right) + \frac{1}{2}Rl_0^2(\cos ec\theta - 1)^2 \qquad \ldots(\text{ii})$$

from (i) and (ii): $v = \sqrt{\frac{19\,gl_0}{32}}$

Example: A block of mass M slides down a plane of angle θ. The problem is to find the speed of the block after it has descended through height h, assuming that it starts from rest and that the coefficient of friction μ is constant.

Initially the block is at rest at height h; finally the block is moving with speed v at height 0. Hence



Solution: The potential energy, kinetic energy and total energy at point a and b respectively

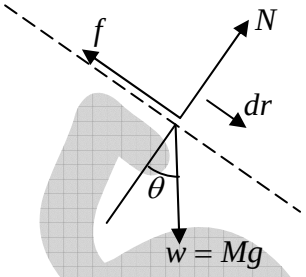
$$U_a = Mgh,\ U_b = 0\ \ K_a = 0\ \ ,\ K_b = \frac{1}{2}Mv^2,\ E_a = Mgh\ \ ,E_b = \frac{1}{2}Mv^2$$

The non-conservative force is $f = \mu N = \mu Mg\cos\theta$. Hence, conservative work is

$$W_{ba}^{nc} = \int_a^b f\cdot dr = -fs,$$

where s is the distance the block slides. The negative sign arises because the direction of f is always opposite to the displacement, so that $f \cdot dr = -f\,dr$. Using $s = h/\sin\theta$, we have

$$W_{ba}^{nc} = -\mu Mg\cos\theta\frac{h}{\sin\theta} = -\mu\cot\theta Mgh.$$



The energy equation $E_b - E_a = W_{ba}^{nc}$ becomes

$$\frac{1}{2}Mv^2 - Mgh = -\mu\cot\theta Mgh,$$

which gives $v = \left[2\left(1-\mu\cot\theta\right)gh\right]^{1/2}$

example prove that for $F = A\left(xy\hat{i} + y^2\hat{j}\right)$ is non conservative and

(b) Consider the integral $W = \int \vec{F}.dr$ from $(0,0)$ to $(0,1)$, first along path 1 and then along path 2, as shown in the figure. Check whether is path independent or not.

(a) $\vec{\nabla}\times\vec{F} = -x\hat{k} \neq 0$. So force is non conservative.

(b) The force F has no physical significance, but the example illustrates the properties of non conservative forces. Since the segments of each path lie along a coordinate axis, it is particularly simple to evaluate the integrals. For path 1 we have

$$\oint_1 F\cdot dr = \int_a F\cdot dr + \int_b F\cdot dr + \int_c F\cdot dr$$

Along segment $adr = dx\hat{i}, F\cdot dr = F_x dx = Axydx$. Since $y = 0$ along the line of this integration,

$\int_a F\cdot dr = 0$. Similarly, for path b, $\int_b F\cdot dr = A\int_{x=1,\,y=0}^{x=1,\,y=0} y^2 dy = \frac{A}{3}$,

while for path c,

$$\int_c F\cdot dr = A\int_{x=1,\,y=1}^{x=1,\,y=1} xy\,dx = A\int_1^0 x\,dx = -\frac{A}{2}$$

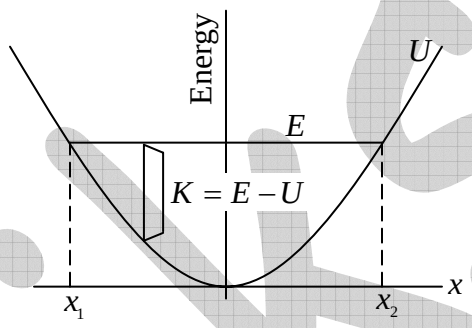
Thus $\oint_1 F\cdot dr = \frac{A}{3} - \frac{A}{2} = -\frac{A}{6}$

Along path 2 we have $\oint_2 F\cdot dr = A\int_{0,0}^{0,1} y^2 dy = \frac{A}{3} \neq \oint_1 F\cdot dr$

2.5 Energy Diagrams

We can often find the most interesting features of the motion of a one dimensional system by using an energy diagram, in which the total energy E and the potential energy U are plotted as functions of position. The kinetic energy $K = E - U$ is easily found by inspection. Since kinetic energy can never be negative, the motion of the system is constrained to regions where $U \le E$.

2.5.1 Energy Diagram of Bounded Motion



Here is the energy diagram for a harmonic oscillator. the potential energy $U = kx^2/2$ is a parabola centered at the origin. Since the total energy is constant for a conservative system, E is represented by a horizontal straight line. Motion is limited to the shaded region where $E \ge U$; the limits of the motion, x_1 and x_2 in the sketch, are sometimes called the turn points.

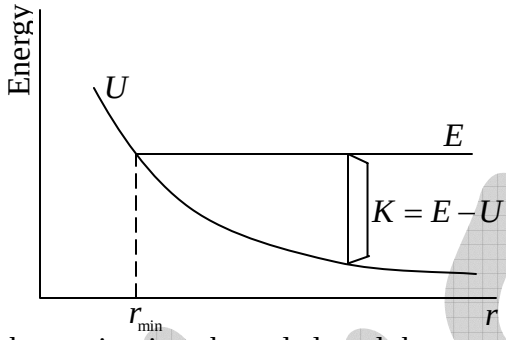
Here is what the diagram tells us. The kinetic energy, $K = E - U$ is greatest at the origin. As the particle flies past the origin in either direction, it is slowed by the spring and comes to a complete rest at one of the turning points x_1, x_2. The particle then moves toward the origin with increasing kinetic energy and the cycle is repeated.

The harmonic oscillator provides a good example of bounded motion. As E increases, the turning points move farther and farther off, but the particle can never move away freely. If E decreased, the amplitude of motion decreases, until finally for $E = 0$ the particle lies at rest at $x = 0$.

2.5.2 Energy Diagram of Unbounded Motion

the pot $U = A/r$, where A is positive. There is a distance of closest approach r_{min}, as shown in the diagram, but the motion is not bounded for large r since U decreases with distance. If the particle is shot toward the origin, it gradually loses kinetic energy until it comes momentarily to

rest at r_{min}. The motion then reverses and the particle moves out toward infinity. The final and initial speeds at any point are identical; the collision merely reverses the velocity.



For positive energy, $E > 0$, the motion is unbounded, and the atoms are free to fly apart. As the diagram indicates, the distance of closest approach, r_{min}, does not change appreciably as E is increased. The steep slope of the potential energy curve at small r .the kinetic energy will zero at r_{min} and as r increases the potential energy decrease but kinetic energy $K = E - U$ will increases sharply .

Phase Curve: The curve between momentum and its conjugate momentum for given value of energy is known as phase curve .

2.6 Stability and Unstability in One Dimension

Any potential can be function of and generalized velocity $\mathrm{U} \equiv \mathrm{U}(x, \dot{x}, t)$ the equilibrium point is defined where total external force on the system is zero i,e for any co-ordinate say x is said to be equilibrium point if $\frac{\partial U}{\partial x} = 0$ at $x = x_0$

If x_0 is maxima or (local maxima) i.e. $\left.\frac{\partial^2 U}{\partial x^2}\right|_{x=x_0} \leq 0$ then it is unstable equilibrium point. unstable equilibrium point always behave like repulsive point.

and If x_0 is minima or (local minima) i.e. $\left.\frac{\partial^2 U}{\partial x^2}\right|_{x=x_0} \geq 0$ then it is stable equilibrium point. Stable equilibrium always behaves as attractive point. The phase portrait is a geometrical representation of trajectories of dynamical system in phase plane which is defined by generalized coordinate and its conjugate generalized momentum. The phase trajectory about unstable point is mainly in

bounded but phase trajectory about stable point is bounded and motion is defined as small oscillation.

Example: If potential in one dimension is given by $V(x) = -\frac{x^2}{2} + \frac{x^4}{4}$ then

(a) Find the point where potential is zero

(b) Find the equilibrium point.

(c) Find the stable and unstable equilibrium

(e) Draw phase curve ie $p(x)$ Vs x for given energy E.

Solution: (a) $V(x) = 0 \Rightarrow -\frac{x^2}{2} + \frac{x^4}{4} = 0 \Rightarrow x = 0, +\sqrt{2}, -\sqrt{2}$

(b) For equilibrium point $\frac{\partial V}{\partial x} = 0 \Rightarrow -x + x^3 = 0$ so there is three equilibrium points

$$x_1 = 0,\ x_2 = 1,\ x_3 = -1$$

(c) For discussion of stability and instability we must found $\frac{\partial^2 V}{\partial x^2} = -1 + 3x^2$ for stable equilibrium $\frac{\partial^2 V}{\partial x^2} > 0$ at $x_2 = 1$ and $x_3 = -1$ the value of $\frac{\partial^2 V}{\partial x^2} = 2$ which is greater than 0 for unstable point $\frac{\partial^2 V}{\partial x^2} < 0$ at $x_1 = 0$ the value of $\frac{\partial^2 V}{\partial x^2} = -1$ which is lesser than 0 so it is unstable point .

(d) $p(x)$ vs x

To plot phase curve first one should plot potential (V vs x), then on same axis one should plot momentum plot with common x axis .

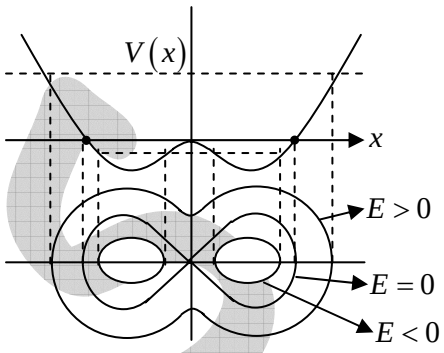
We can check how momentum is changing with position with keeping in mind how potential is changing

With position .for example if potential is increasing then kinetic will decreasing or vice versa because total energy will always remain constant. One should plot the phase curve for different range of energy. For example in this potential there are three range of energy

Case 1- If $-\frac{1}{4} < E < 0$ the particle has motion about stable equilibrium point $x = 1, -1$ the motion is bounded .

Case 2- If $0 < E < \infty$ the particle has motion about unstable equilibrium point $x = 0$ the motion is bounded .

Case 3- At $E = 0$ the particle can be landed exactly at unstable equilibrium which is nature of transition from case-1 to case- 2.



2.6.1 Small Oscillation:

Let us assume the potential $U(x)$ has stable equilibrium point at $x = x_0$ then $\left.\frac{\partial U}{\partial x}\right|_{x=x_0} = 0$ and $\left.\frac{\partial^2 U}{\partial x^2}\right|_{x=x_0} \geq 0$

one will do Taylor expansion of $U(x)$ about $x = x_0$, then

$$U(x) = U(x_0) + \left.\frac{\partial U}{\partial x}\right|_{x=x_0}(x - x_0) + \frac{1}{2}\left.\frac{\partial^2 U}{\partial x^2}\right|_{x=x_0}(x - x_0)^2 + order(x - x_0)^3 .....$$

If term $(x - x_0)^2$ is small then higher order term can be neglected then potential energy is equivalent to $U(x) = U(x_0) + \frac{1}{2}\left.\frac{\partial^2 U}{\partial x^2}\right|_{x=x_0}(x - x_0)^2$ because $\left.\frac{\partial U}{\partial x}\right|_{x=x_0} = 0$

So force is equal to $F = -\frac{\partial U}{\partial x}$, $F = -\left.\frac{\partial^2 U}{\partial x^2}\right|_{x=x_0}(x - x_0)$. Hence $F \propto -(x - x_0)$ then motion is small oscillation the angular frequency is given by $\omega = \sqrt{\frac{\left.\frac{\partial^2 U}{\partial x^2}\right|_{x=x_0}}{m}}$ where m is mass of the particle . the term $k = \left.\frac{\partial^2 U}{\partial x^2}\right|_{x=x_0}$ is identified as spring constant.

Example: If particle of mass m is interact with potential $ax^2 + \frac{b}{x^2}$ then what is frequency of the oscillation? (Assume oscillation is small)?

Solution: $V(x) = ax^2 + \frac{b}{x^2}$

For equilibrium point $\frac{\partial V}{\partial x} = 0$

$$2ax - \frac{2b}{x^3} = 0 \quad \Rightarrow ax^4 - b = 0 \Rightarrow x_0 = \left(\frac{b}{a}\right)^{1/4}$$

$$k = \left.\frac{\partial^2 V}{\partial x^2}\right|_{x=x_0} = 2a + \frac{2.3b}{x^4} \qquad \text{Put value of } x_0 = \left(\frac{b}{a}\right)^{1/4}$$

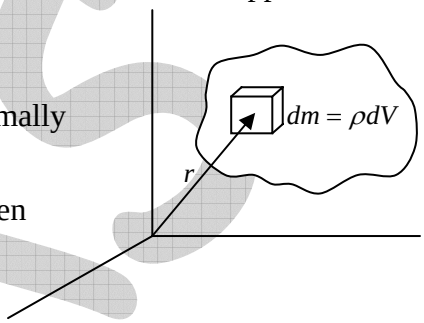
$$k = 2a + \frac{6b \times a}{b} = 8a \;\Rightarrow \omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{8a}{m}}$$

2.7 Center of Mass

N mass elements. If r_j is the position of the jth element, and m_j is its mass, then center of mass is defined as $\vec{R} = \frac{1}{M}\sum_{j=1}^{N} m_j \vec{r}_j$

2.7.1 Center of Mass of Continuous System

The result is not rigorous, since the mass elements are not true particles. However, in the limit where N approaches infinity, the size of each element approaches zero and the approximation becomes exact.



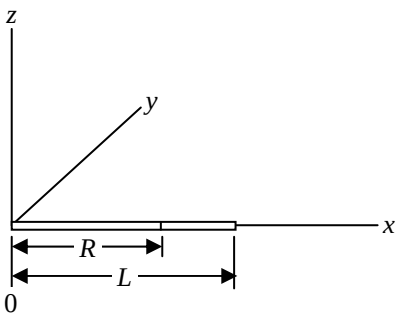
$\vec{R} = \lim_{N\to\infty} \frac{1}{M}\sum_{j=1}^{N} m_j \vec{r}_j$. This limiting process defines an integral. Formally $\lim_{N\to\infty} \sum_{j=1}^{\infty} m_j r_j = \int r\, dm$, where dm is a differential mass element. Then

$$\vec{R} = \frac{1}{M}\int \vec{r}\, dm .$$

To visualize this integral, think of dm as the mass in an element of volume dV located at position r. If the mass density at the element is ρ, then $dm = \rho\, dV$ and $\vec{R} = \frac{1}{M}\int \vec{r}\,\rho dV$. This integral is called a volume integral.

Example: A rod of length L has a non-uniform density. λ, the mass per Unit length of the rod, varies as $\lambda = \lambda_0 (s/L)$ where is a constant and s is the distance from the end marked 0. Find the center of mass.

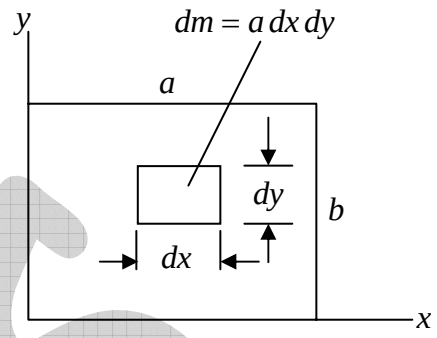
Solution: It is apparent that R lies on the rod. Let the origin of the coordinate system coincide with the end of the rod, 0, and let the x axis lie along the rod so that $s = x$. The mass in an element of length dx is $dm = \lambda, dx = \lambda_0 x\, dx / L$. The rod extends from $x = 0$ to $x = L$ and the total mass is $M = \int dm = \int_0^L \lambda\, dx = \int_0^L \frac{\lambda_0 x dx}{L} = \frac{1}{2}\lambda_0 L$



The center of mass is at $\vec{R} = \frac{1}{M}\int r\lambda dM = \frac{2}{\lambda_0 L}\int_0^L \left(x\hat{i} + 0\hat{j} + 0\hat{k}\right)\frac{\lambda_0 x dx}{L} = \frac{2}{L^2}\frac{\hat{i}}{3}x^3\Big|_0^L = \frac{2}{3}L\hat{i}$

Example: Find the center of mass of a thin rectangular plate with sides of length a and b, whose mass per unit area σ varies in the following fashion:

$\sigma = \sigma_0 (xy/ab)$, where σ_0 is a constant.



Solution: $\vec{R} = \frac{1}{M}\iint \left(x\hat{i} + y\hat{j}\right)\sigma dx\, dy$,

We find M, the mass of the plate, as follows:

$$M = \int_0^b \int_0^a \sigma dx\, dy = \int_0^b \int_0^a \sigma \frac{x}{a}\frac{y}{b} dx\, dy$$

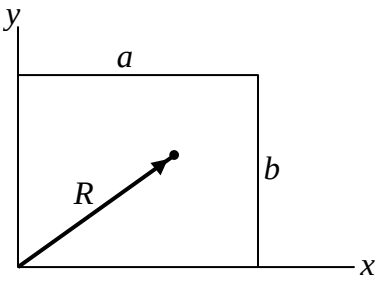
We first integrate over x, treating y as a constant.

$$M = \int_0^b \left(\int_0^a \sigma_0 \frac{x}{a}\frac{y}{b} dx \right) dy = \int_0^b \left(\sigma_0 \frac{y}{b}\frac{x^2}{2a}\Big|_{x=0}^{x=a} \right) dy = \int_0^b \sigma_0 \frac{y}{b}\frac{a}{2} dy = \frac{\sigma_0 a}{2}\frac{y^2}{2b}\Big|_{y=0}^{y=b} = \frac{1}{4}\sigma_0 ab.$$

The x component of R is

$$X = \frac{1}{M}\iint xa\, dx\, dy = \frac{1}{M}\int_0^b \left(\int_0^a x\sigma_0 \frac{xy}{ab} dx \right) dy = \frac{1}{M}\int_0^b \left(\frac{\sigma_0 y}{ab}\frac{x^3}{3}\Big|_0^a \right) dy$$

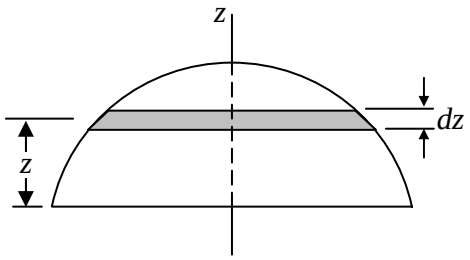
$$= \frac{1}{M}\frac{\sigma_0}{ab}\int_a^b \frac{ya^3}{3} dy = \frac{1}{M}\frac{\sigma_0}{ab}\frac{a^3}{3}\frac{b^2}{2} = \frac{4}{\sigma_0 ab}\frac{\sigma_0 a^2 b}{6} = \frac{2}{3}a$$



Similarly, $Y = \frac{2}{3}b$.

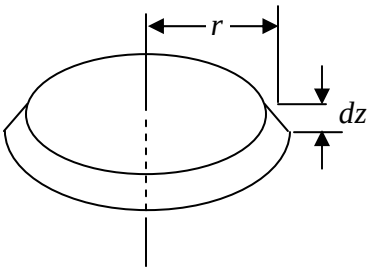
So center of mass is $\vec{R} = \frac{2}{3}(a\hat{i} + b\hat{j})$

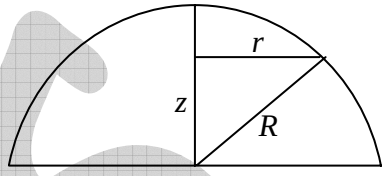
Example: Find the center of mass of a uniform solid hemisphere of radius R and mass M. From symmetry it is apparent that the center of mass lies on the z axis, as illustrated. Its height above the equatorial plane is



$$Z = \frac{1}{M}\int z dM.$$

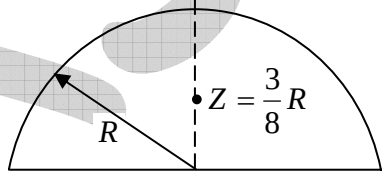
Solution: The integral is over three dimensions, but the symmetry of the situations lets us treat it as a one dimensional integral. We mentally subdivide the hemisphere into a pile of thin disks. Consider the circular disk of radius r and thickness dz. Its volume is $dV = \pi r^2 dz$, and its mass is $dM = \rho dV = (M/V)(dV)$, where $V = \frac{2}{3}\pi R^3$. Hence,





$$Z = \frac{1}{M}\int \frac{M}{V} z dV = \frac{1}{V}\int_{z=0}^{R} \pi r^2 z\, dz$$

To evaluate the integral we need to find r in terms of z. Since $r^2 = R^2 - z^2$, we have



$$Z = \frac{\pi}{V}\int_0^R z\left(R^2 - z^2\right) dz = \frac{\pi}{V}\left(\frac{1}{2}z^2R^2 - \frac{1}{4}z^4\right)\Big|_0^R$$

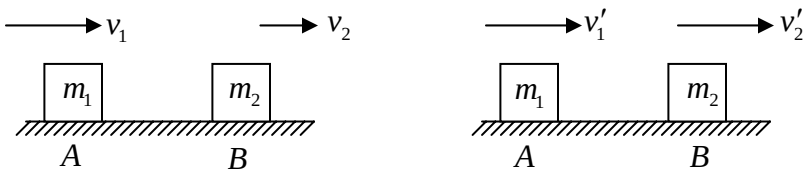
$$= \frac{\pi}{V}\left(\frac{1}{2}R^4 - \frac{1}{4}R^4\right) = \frac{\frac{1}{4}\pi R^4}{\frac{2}{3}\pi R^3} = \frac{3}{8}R.$$

2.8 Conservation of Momentum

In the last section we found that the total external force F acting on a system is related to the total momentum P of the system by $F = \frac{dP}{dt}$

Consider the implications of this for an isolated system, that is, a system which does not interact with its surroundings. In this case $F = 0$, and $dP/dt = 0$. The total momentum is constant; no matter how strong the interactions among an isolated system of particles, and no matter how complicated the motions, the total momentum of an isolated system is constant. This is the law of conservation of momentum. As we shall show, this apparently simple law can provide powerful insights into complicated systems.

2.8.1 Elastic Collision in One Dimension



Consider two elastic bodies A and B moving along the same line (figure). The body A has a mass m_1 and moves with a velocity v_1 towards right and the body B has a mass m_2 and moves with a velocity v_2 in the same direction. We assume $v_1 > v_2$ so that the two bodies may collide. Let v_1' and v_2' be the final velocities of the bodies after the collision. The total linear momentum of the two bodies remains constant, so that,

$$m_1v_1 + m_2v_2 = m_1v_1' + m_2v_2' \qquad \text{(i)}$$

or $$m_1v_1 - m_1v_1' = m_2v_2' - m_2v_2$$

or $$m_1\left(v_1 - v_1'\right) = m_2\left(v_2' - v_2\right) \qquad \text{(ii)}$$

Also, since the collision is elastic, the kinetic energy before the collision is equal to the kinetic energy after the collision. Hence,

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1v_1'^2 + v_2'^2$$

or, $$m_1v_1^2 - m_1v_1'^2 = m_2v_2'^2 - m_2v_2^2$$

or, $$m_1\left(v_1^2 - v_1'^2\right) = m_2\left(v_2'^2 - v_2^2\right) \qquad \text{(iii)}$$

Dividing (iii) by (ii)

$$v_1 + v_1' = v_2' + v_2$$

$$v_1 - v_2 = v_2' - v_1' \qquad \text{(iv)}$$

Now, $\left(v_1 - v_2\right)$ is the rate at which the separation between the bodies decreases before the collision. Similarly, $\left(v_2' - v_1'\right)$ is the rate of increase of separation after the collision. So the equation (iv) may be written as

Velocity of separation (after collision) = Velocity of approach (before collision)...

This result is very useful in solving problems involving elastic collision. The final velocities v_1' and v_2' may be obtained from equation (i) and (iv). Multiply equation (iv) by m_2 and subtract from equation (i).

$$2m_2v_2+(m_1+m_2)v_1'$$

or $$v_1'=\frac{(m_1-m_2)}{m_1+m_2}v_1+\frac{2m_2}{m_1+m_2}v_2 \qquad (1)$$

Now multiply equation (iv) by m_1 and add to equation (i),

$$2m_1v_1-(m_1-m_2)v_2=(m_2+m_1)v_2'$$

or $$v_2'=\frac{2m_1v_1}{m_1+m_2}-\frac{(m_1-m_2)v_2}{m_1+m_2} \qquad (2)$$

Equations, (1) and (2) give the final velocities in terms of the initial velocities and the masses.

Special Cases:

(a) Elastic collision between a heavy body and a light body:

Let $m_1>>m_2$. A heavy body hits a light body from behind.

We have, $$\frac{m_1-m_2}{m_1+m_2}\approx 1,\ \frac{2m_2}{m_1+m_2}\approx 0 \text{ and } \frac{2m_1}{m_1+m_2}\approx 2$$

With these approximations the final velocities of the bodies are, from (1) and (2),

$$v_2'\approx v_1 \text{ and } v_2'\approx 2v_1-v_2.$$

The heavier body continues to move with almost the same velocity, If the lighter body were kept at rest $v_2=0, v_2'=2v_1$ which means the lighter body, after getting a push from the heavier body will fly away with a velocity double the velocity of the heavier body.

Next suppose $m_2>>m_1$. A light body hits a heavy body from behind.

We have,

$$\frac{m_1-m_2}{m_1+m_2}\approx -1,\ \frac{2m_2}{m_1+m_2}\approx 2 \text{ and } \frac{2m_1}{m_1+m_2}\approx 0$$

The final velocities of the bodies are, from (1) and (2), $v_1'\approx -v_1+2v_2$ and $v_1'\approx v_2$

The heavier body continues to move with almost the same velocity, the velocity of the lighter body changes. If the heavier body were at rest, $v_2 = 0$ then $v_1' = -v_1$ the lighter body returns alter collision with almost the same speed. This is the case when a ball collides elastically with a fixed wall and returns with the same speed.

(b) Elastic collision of two bodies of equal mass:

Putting $m_1 = m_2$ in equation (1) and (2) $v_1' = v_2$ and $v_2' = v_1$.

When two bodies of equal mass collide elastically, their velocities are mutually interchanged.

2.8.2 Perfectly Inelastic Collision in One Dimension

Final Velocity

When perfectly inelastic bodies moving along the same line collide, they stick to each other. Let m_1 and m_2 be the masses, v_1 and v_2 be their velocities before the collision and V be the common velocity of the bodies after the collision. By the conservation of linear momentum,

$$m_1v_1 + m_2v_2 = m_1V + m_2V$$

or $$V = \frac{m_1v_1 + m_2v_2}{m_1 + m_2} \qquad \text{(i)}$$

2.8.3 Elastic Collision in Two Dimensions

Consider two objects A and B of mass m_1 and m_2 kept on the X - axis (figure). Initially, the object B is at rest and A moves towards B with a speed u_1. If the collision is not head -on (the force during the collision is not along the initial velocity), the objects move along different lines. Suppose the object A moves with a velocity $\vec{v}$ making an angle θ with the X - axis and the object B moves with a velocity $\vec{v}_2$ making an angle ϕ with the same axis. Also, suppose $\vec{v}_1$ and $\vec{v}_2$ lie in X - Y plane. Using conservation of momentum in X and Y directions, we get

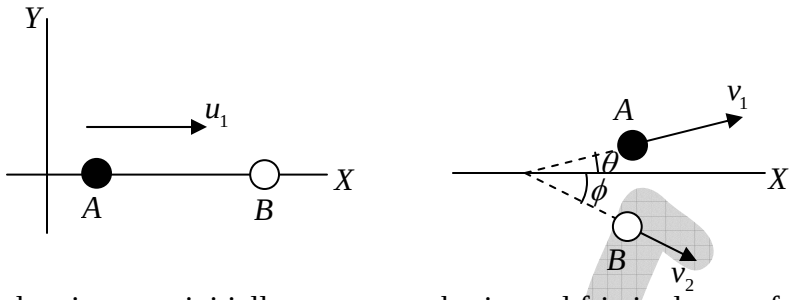
$$m_1u_1 = m_1v_1\cos\theta + m_2v_2\cos\theta \qquad \text{(i)}$$

and $$0 = m_1v_1\sin\theta - m_2v_2\sin\phi \qquad \text{(ii)}$$

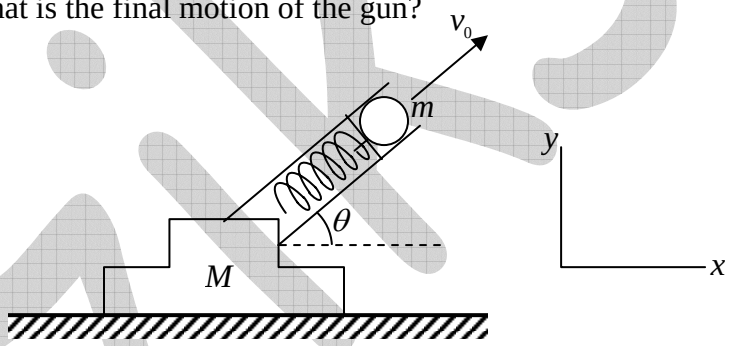
If the collision is elastic, the final kinetic energy is equal to the initial kinetic energy. Thus,

$$\frac{1}{2}m_1u_1^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 \qquad \text{(iii)}$$

We have four unknowns v_1, v_2, θ and ϕ to describe the final motion whereas there are only three relations. Thus, the final motion cannot be uniquely determined with this information.



Example: A loaded spring gun, initially at rest on a horizontal frictionless surface, fires a marble at angle of elevation θ. The mass of the gun is M, the mass of the marble is m, and the muzzle velocity of the marble is v_0. What is the final motion of the gun?

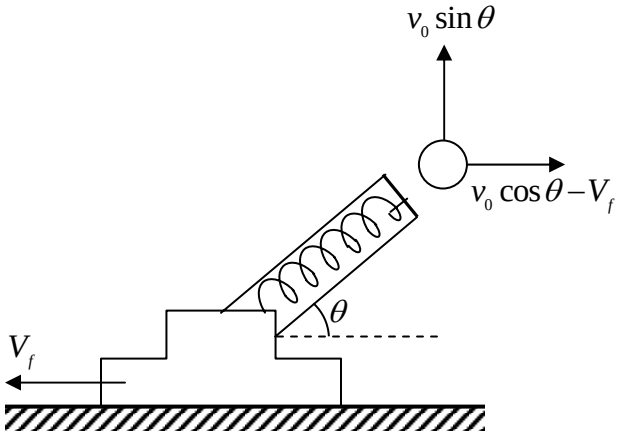


Solution: Take the physical system to be the gun and marble, Gravity and the normal force of the table act on the system. Both these forces are vertical. Since there are no horizontal external forces the x component of the vector equation $F = dP/dt$ is

$$0 = \frac{dP_x}{dt} \qquad (1)$$

According to equation (1) P_x is conserved:

$$P_{x,\text{initial}} = P_{x,\text{final}} \qquad (2)$$



Head office
fiziks, H.No. 23, G.F, Jia Sarai,
Near IIT, Hauz Khas, New Delhi-16
Phone: 011-26865455/+91-9871145498

Branch office
Anand Institute of Mathematics,
28-B/6, Jia Sarai, Near IIT
Hauz Khas, New Delhi-16

Website: www.physicsbyfiziks.com
Email: fiziks.physics@gmail.com

Let the initial time be prior to firing the gun. Then $P_{x,\text{initial}} = 0$. Since the system is initially at rest. After the marble has left the muzzle, the gun recoils with some speed V_f, and its final horizontal momentum is MV_f, to the left. Finding the final velocity of the marble involves a subtle point, however. Physically, the marble's acceleration is due to the force of the gun, and the gun's recoil is due t the reaction force of the marble. The gun stops accelerating once the marble leaves the barrel, so that at the instant the marble and the gun part company, the gun has its final speed V_f. At that same instant the speed of the

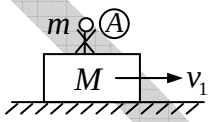
marble relative to the gun is v_0. Hence, the final horizontal speed of the marble relative to the table is $v_0 \cos\theta - V_f$. By conservation of horizontal momentum we therefore have

$$0 = m\left(v_0 \cos\theta\right) - V_f - MV_f$$

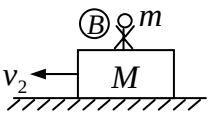
or $$V_f = \frac{mv_0 \cos\theta}{M+m}$$

Example: Two identical buggies 1 and 2 with one man in each move without friction due to inertia along the parallel rails toward each other. When the buggies get opposite each other, the men exchange their places by jumping in the direction perpendicular to the motion direction. As a consequence, buggy 1 stops and buggy 1 keeps moving in the same direction, with its velocity becoming equal to v. Find the initial velocities of the buggies v_1 and v_2 if the mass of each buggy (without a man) equals M and the mass of each man m.

Solution: status of buggies just before jump

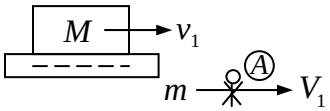


Buggy (1)

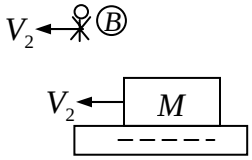


Buggy (2)

status of buggies just during jump

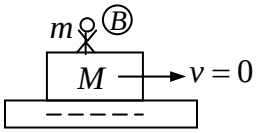


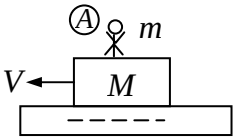
Buggy (1)



Buggy (2)

status of buggies after jump





Buggy (1) Buggy (2)

During this exchange momentum will be conserved because there is no force is horizontal direction.

Conservation of momentum for buggy (2) $Mv_1 - mv_2 = 0$ …(i)

Conservation of momentum for buggy (2)

$$MV_2 - mV_1 = (m+M)V \qquad \text{…(ii)}$$

From (i) and (ii):

$$V_2 = \frac{MV}{M-m} \quad \Big| \quad V_1 = \frac{mV}{M-m}$$

But in term of vector: V_2 has opposite direction as V_1.

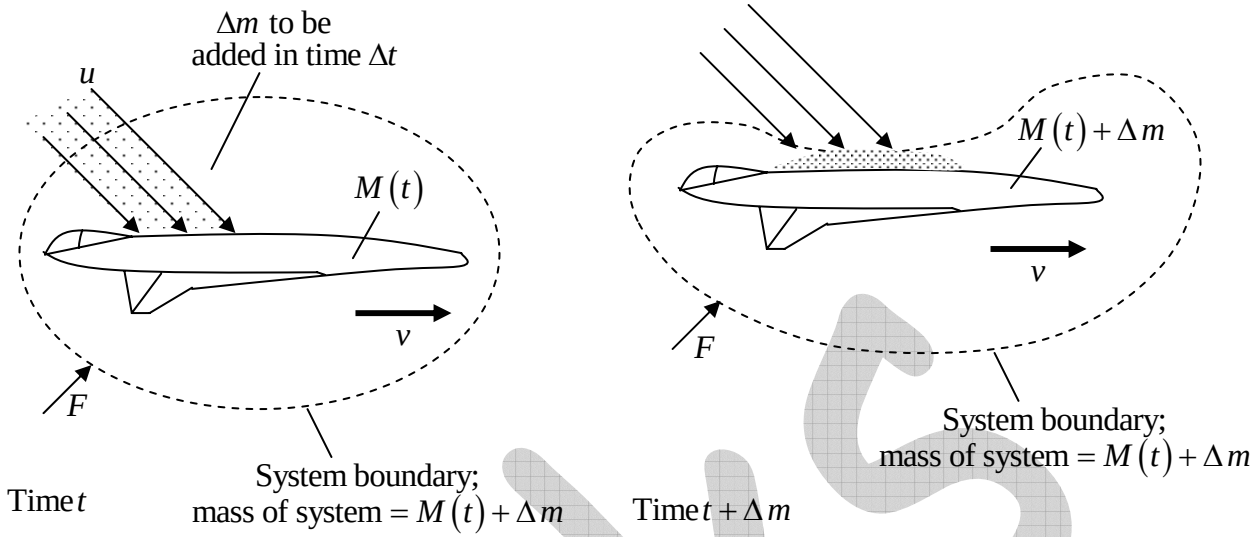
Then $V_2 = \dfrac{MV}{M-m}$ and $V_1 = \dfrac{-mV}{M-m}$

2.9 Variable Mass and Momentum

Case 1-Motion of Spacecraft.

A spacecraft moves through space with constant velocity v. The space craft encounters a stream of dust particles which embed themselves in it at rate dm/dt. The dust has velocity u just before it hits. At time t the total mass of the spacecraft is $M(t)$. The problem is to find external force F necessary to keep the spacecraft moving uniformly. (In practice F would most likely come from the spacecraft's own rocket engines. For simplicity, we can visualize the source F to be completely external – an invisible hand, so to speak.)

Let us focus on the short time interval between t and $t+\Delta t$. The drawings below show the system at the beginning and end of the interval.



Let Δm denote the mass added to the satellite during Δt. The system consists of $M(t)$ and Δm. The initial momentum is $P(t) = M(t)v + (\Delta m)u$.

The final momentum is $P(t+\Delta t) - P(t) = (v-u)\Delta m$.

The rate of change of momentum is approximately

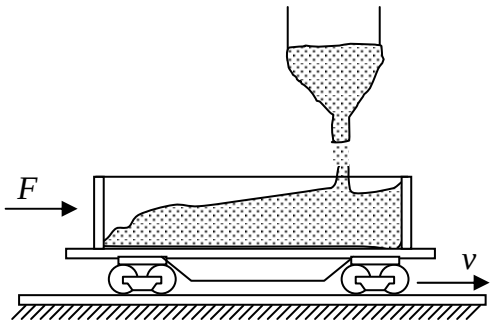
$$\frac{\Delta P}{\Delta t} = (v-u)\frac{\Delta m}{\Delta t}.$$

In the limit $\Delta t \to 0$, we have the exact result $\frac{dP}{dt} = (v-u)\frac{dm}{dt}$.

Since $F = dP/dt$, the required external force is $F = (v-u)\frac{dm}{dt}$.

Note that F can be either positive or negative depending on the direction of the stream of mass. If $u = v$, the momentum of the system is constant, and $F = 0$.

Example: Sand falls from a stationary hopper onto a freight car which is moving with uniform velocity v. The sand falls at the rate dm/dt. How much forces needed to keep the freight car moving a the speed v?



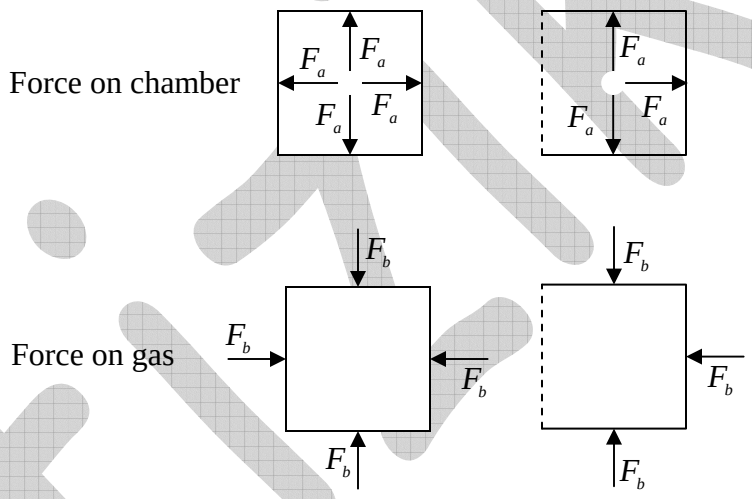
Solution: In this case, the initial speed of the sand is 0, and

$$\frac{dP}{dt} = (v-u)\left(\frac{dm}{dt}\right) = v\frac{dm}{dt}$$

The required force is $F = v\dfrac{dm}{dt}$. We can understand why this force is needed by considering in detail just what happens to a sand grain as it lands on the surface of the freight car.

Case -2 Motion of Rocket

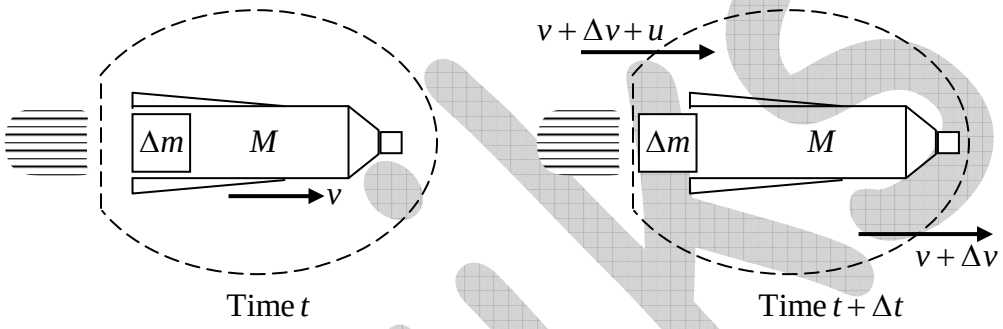
The concept of momentum is invaluable in understanding the motion of a rocket. A rocket accelerates by expelling gas at a high velocity; the reaction force of the gas on the rocket accelerates the rocket in the opposite direction. The mechanism is illustrated by the drawings of the cubical chamber containing gas at high pressure



The gas presses outward on each wall with the force F_a (We show only four walls for clarity.) The vector sum of the $F_a's$ is zero, giving zero net force on the chamber. Similarly each wall of the chamber exerts a force on the gas $F_b = -F_a$; the net force on the gas is also zero. In the right hand drawings below, one wall has been removed. The net force on the chamber is F_a, to the right. The net force on the gas is F_b, to the left. Hence the gas accelerates to the left, and the chamber accelerates to the right.

To analyze the motion of the rocket in detail, we must equate the external force on the system, F, with the rate of change of momentum, dP/dt. Consider the rocket at time t. Between t and $t+\Delta t$ a mass of fuel Δm is burned and expelled as gas with velocity u relative to the rocket. The exhaust velocity u is determined by the nature of the propellants, the throttling of the engine, etc., but it is independent of the velocity of the rocket.

The sketches below show the system at time t and at time $t+\Delta t$. The system consists of Δm plus the remaining mass of the rocket M. Hence the total mass is $M+\Delta m$.



The velocity of the rocket at time t is $v(t)$, and at $t+\Delta t$ it is $v+\Delta v$.

The initial momentum is $P(t)=(M+\Delta m)$

the final momentum is $P(t+\Delta t)=M(v+\Delta v)+\Delta m(v+\Delta v+u)$

The change in momentum is $\Delta P=P(t+\Delta t)-P(t)=M\Delta v+(\Delta m)u$ assume $\Delta m.\Delta v=0$

Therefore, $\frac{dP}{dt}=\lim_{\Delta t\to 0}\frac{\Delta P}{\Delta t}=M\frac{dv}{dt}+u\frac{dm}{dt}$.

Note that we have defined u to be positive in the direction of v. In most rocket applications, u is negative, opposite to v. It is inconvenient to have both m and M in the equation. dm/dt is the rate of increase of the exhaust mass. Since this mass comes from the rocket,

$$\frac{dm}{dt}=-\frac{dM}{dt}.$$

equating the external force to dP/dt, we obtain the fundamental rocket equation

$$F=M\frac{dv}{dt}-u\frac{dM}{dt}.$$

It may be useful to point out two minor subtleties in our development. The first is that the velocities have been expressed with respect to an inertial frame, not a frame attached to the rocket. The second is that we took the final velocity of the element exhaust gas to be $v+\Delta v+u$ rather than $v+u$.

Example: (a) A rocket is moving vertically in absence of gravity .if initial exhaust speed and mass of rocket is u and M_0 respectively. What will be speed if mass became αM_0 after time $t=t_f$. (assume $t=0$ velocity of rocket is v_0.)

(b) How situation will different if a gravitational field is present.

Solution: (a) If there is no external force on a rocket, $F=0$ and its motion is given by

$$M\frac{dv}{dt}=u\frac{dM}{dt} \qquad \text{or} \qquad \frac{dv}{dt}=\frac{u}{M}\frac{dM}{dt}$$

Generally the exhaust velocity i is constant, in which case it is easy to integrate the equation of motion.

$$\int_{t_b}^{t_f}\frac{dv}{dt}dt=u\int_{t_b}^{t_f}\frac{1}{M}\frac{dM}{dt}dt=u\int_{M_0}^{M_f}\frac{dM}{M}$$

or $$v_f-v_0=u\ln\frac{M_f}{M_0}=u\ln\alpha$$

(b) If a rocket takes off in a constant gravitational field,

$M_g=M\frac{dv}{dt}-u\frac{dM}{dt}$, where u and g are directed down and are assumed to be constant.

$$\frac{dv}{dt}=\frac{u}{M}\frac{dM}{dt}+g$$

Integrating with respect to time, we obtain

$$v_f-v_0=u\ln\left(\frac{M_f}{M_0}\right)+g\left(t_f-t_0\right)\Rightarrow v_f=v_0+u\ln(\alpha)+gt_f.$$

MCQ (Multiple Choice Questions)

Q1. Match the following List I with List II.

List I	List II
A. Elastic collision	1. Momentum constant but K.E. is not constant
B. Ineslastic collision	2. Moment and K.E. is constant.
C. Perfectly inelastic collision	3. $0 < e < 1$
D. Partly elastic collision	4. $e = o$

	A	B	C	D
(a)	2	4	3	1
(b)	2	1	4	3
(c)	1	2	3	4
(d)	4	3	1	2

Q2. Two bodies of mass $100\ g$ and $20\ g$ are moving with velocities $\left(2\hat{i}-7\hat{j}+3\hat{k}\right)$ and $\left(-10\hat{i}+35\hat{j}+3\hat{k}\right) cm/s$ respectively. The velocity of the centre of mass is given by .

(a) $2k\ m/s$ (b) $\left(-8\hat{i}+28\hat{j}\right) cm/s$

(c) $\left(\frac{10}{2}\hat{i}-\frac{35}{6}\hat{j}+3\hat{k}\right) cm/s$ (d) $2\hat{k}\ cm/s$

Q3. A $2\,kg$ mass is acted upon by a force which gives a time dependent displacement of $x=\frac{t^2}{2}$. The work done in the time interval 0 to t_0 is

(a) $0.5t_0$ (b) t_0 (c) t_0^2 (d) t_0^3

Q4. A body of mass m, moving with velocity u collides elastically with another body at rest having mass M. If the body of mass M moves with velocity v, then the velocity of the body of mass m after the impact is

(a) $\frac{m-M}{m+M}u$ (b) $\frac{m-M}{m+M}v$ (c) $\frac{mu+Mv}{m+M}$ (d) $\frac{mu-Mv}{m+M}$

Q5. The mass of a rocket is M and the total mass of rocket and the fuel is M_0. The average exhaust velocity of gases ejected from rocket motors is u are the final velocity attained by the rocket after using all fuel is v. The final velocity v is proportional to

(a) $\log\left(\frac{M_0}{M}\right)$ (b) $\left(\frac{Mu}{M_0}\right)$

(c) $\left(M_0 - M\right)$ (d) $\left(M_0 - M\right)^{-1}$

Q6. A block of mass $2\ kg$ is attached with a spring of spring constant $4900\ Nm^{-1}$ and the system is kept on smooth, horizontal plane the other end of spring is attached with a wall initially, spring is stretched by $5\ cm$ from its natural length and the block is at rest. Now, suddenly an impulse of $4\ kg\,ms^{-1}$ is given to the block towards the wall. The velocity of block, when spring acquires natural length is

(a) $5\ ms^{-1}$ (b) $3\ ms^{-1}$ (c) $6\ ms^{-1}$ (d) None

Q7. Pick out the **WRONG** alternative.

(a) A force is said to be conservative if the work done by the force, when a particle moves in a closed loop, is zero.

(b) If the work done by a force is path independent then the force is said to be conservative.

(c) Work done by frictional force is path independent.

(d) Frictional force is a non-conservative.

Q8. Pick out the correct alternative.

(a) The concept of potential energy can be associated with any type of force.

(b) The mechanical energy of a system is the sum of kinetic potential and internal energy.

(c) The total mechanical energy of a system remains constant if the internal forces are conservative and external forces do no work.

(d) Suppose the internal forces within a system are conservative and external forces do negative work then the mechanical energy of the system increases.

Q9. Pick out the **WRONG** alternative:

(a) When an object of mass m moves from height h_1 above the earth surface to height h_2 the change in its gravitational potential energy is $mg\left(h_2 - h_1\right)$.

(b) A block of mass m slides down a frictionless incline, then the speed of block of the bottom is $\sqrt{2gh}$ where h is the height of incline.

(c) When the free end of a spring is displaced from x_1 to x_2, the change in its potential energy is $\frac{1}{2}K\left(x_2^2 - x_1^2\right)$, where k is the spring constant.

(d) The change in potential energy of a system is equal to the work done by the conservative force on the system.

Q10. A particle is rotated in a vertical circle by connecting it to a spring of length l and keeping to other end of the string fixed. The minimum speed of the particle when the string is horizontal for which the particle will complete the circle is

(a) $\sqrt{gl}$ (b) $\sqrt{2gl}$ (c) $\sqrt{3gl}$ (d) $\sqrt{5gl}$

Q11. Two springs A and $B\left(k_A = 2k_B\right)$ are stretched by applying forces of equal magnitude of the four ends. If the energy stored in A is E, that in B is

(a) $\frac{E}{2}$ (b) $2E$ (c) E (d) $\frac{E}{4}$

Q12. Consider the following two statements

(A) Linear momentum of the system remains constant.

(B) Centre of mass of the system remains at rest.

(a) A implies B and B implies A

(b) A does not imply B and B does not imply A

(c) A implies B and B does not imply A

(d) B implies A but A does not imply B

Q13. A body falling vertically downwards under gravity breaks in two parts of unequal masses. The centre of mass of the two parts taken together shifts horizontally towards

(a) heavier piece

(b) lighter piece

(c) does not shift horizontally

(d) depends on the vertical velocity of the time of breaking

Q14. A body of rest breaks into two pieces of equal masses. The parts will move

(a) in the same direction

(b) along different lines

(c) in opposite directions with equal speeds

(d) in opposite directions with unequal speeds

Q15. Pick out the correct alternative

(a) If the net external force acting on a system is zero, its acceleration is non-zero.

(b) If the velocity of centre of mass is constant, then linear momentum is necessarily constant.

(c) During collision of a system of particles linear momentum is not constant.

(d) Conservation of momentum for a system implies the conservation of energy for that system.

Q16. Two balls are thrown simultaneously in air. The acceleration of the centre of mass of the two balls while in air

(a) depends on the direction of motion of the balls

(b) depends on the masses of two balls

(c) depends on the speed of the two balls

(d) is equal to g

Q17. The potential energy function associated with the force $\vec{F} = 4xy\hat{i} + 2x^2\hat{j}$ is

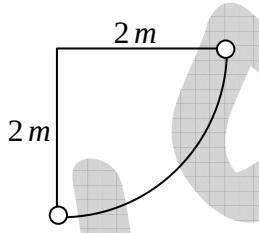
(a) $u = -2x^2y$ (b) $u = -2x^2y + \text{constant}$

(c) $u = 2x^2y + \text{constant}$ (d) Not defined

Q18. When a spring is stretched by $2\,cm$, it stores $100\ J$ of energy. If it is stretched further by $2\,cm$, the stored energy will be increased by

(a) $100\ J$ (b) $200\ J$ (c) $300\ J$ (d) $400\ J$

Q19. The bob of a pendulum of length $2\,m$ lies at P. When it reaches Q, it loses 10% of its total energy due to air resistance. The velocity at Q is (Take $g = 10\,m/s^2$).



(a) $6\,m/s$

(b) $1\ m/s$

(c) $2\ m/s$

(d) $8\ m/s$

Q20. A block moving horizontally on a smooth surface with a speed of $20\ m/s$ breaks into two equal parts continuing in the same direction. If one of the parts moves of $30\ m/s$, with what speed does the second part move

(a) $5\ m/s$ (b) $10\ m/s$ (c) $15\ m/s$ (d) $20\ m/s$

Q21. A car of mass M is moving with a uniform velocity v on a horizontal road when a man drops himself on it from above. Taking the mass of the man to be m, the velocity of the car after the event is

(a) $\frac{3mv}{M+m}$ (b) $\frac{mv}{M+m}$ (c) $\frac{Mv}{M+m}$ (d) $\frac{(M+m)v}{M}$

Q22. You lift a suitcase from the floor and keep it on a table. The work done by you on the suitcase does not depend on

(a) the path taken by the suitcase (b) the time taken by you in doing so

(c) the weight of the suitcase (d) your weight

NAT (Numerical Answer Type)

Q23. $2\ kg$ mass is tied to one end of a string $2\ m$ in length which is hanging vertically. What minimum horizonta1 speed should be imparted to the mass such that it just reaches the top of the vertical circle with equal to the. length of the string? (Take $g = 10\ m/s^2$) _____ m/s

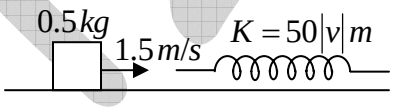
Q24. A $10^4\ kg$ rocket standing on its launch pad can expel exhaust gas at the speed of $2\times10^3\ m/s$ Just to lift the rocket from the ground, the exhaust gas must be ejected at ____ $kg\ s^{-1}$

Q25. A $25\ kg$ mass, starting from rest at the top, slides down a plane that makes an angle of 30^0 with the horizontal. When it reaches the bottom of the $10\ m$ long slide, its velocity is $8\ m/s$. The work done by the force of friction is closest to a value of _____ J

Q26. An object of mass m moving with a velocity v approaching a second object of the same mass but rest. The total kinetic energy of the two objects viewed from the centre of mass is…… αmv^2 the value of α

Q27. Work done when a force $\vec{F} = (\hat{x} + 2\hat{y} + 3\hat{z})$ acting on a particle takes it from the point $\vec{r}_1 = (\hat{x} + \hat{y} + \hat{z})m$ to the point $\vec{r}_2 = (\hat{x} - \hat{y} + 2\hat{z})m$ is _____ J

Q28. A mass of 0.5 kg moving with a speed of 1.5 m/s on a horizontal smooth surface, collides with a weightless spring of force constant $k = 50\ N/m$. The maximum compression of the spring would be_____ m.



Q29. A moving body of mass m and velocity $3\ m/s$ collides with a motionless body of mass $2m$ and sticks to it. Now the velocity of the combined mass would be _____ m/s.

Q30. A block of mass M is attached to the lower end of a vertical spring. The spring is hung from the ceiling and has force constant value k. The mass is released from rest with the spring initially unstretched. The maximum extension produced in the length of the spring will be _____ Mg/k.

Q31. A particle is projected at an angle of 60^0 with the horizontal with a kinetic energy E. The kinetic energy of the highest point is αE, then value of α is _____.

Q32. A uniform chain of length $2\ m$ is kept on a table such that a length of $60\,cm$ hangs freely from the edge of the table. The total mass of the chain is $4\ kg$. The work done in pulling the entire chain on the table is _____ J.

Q33. A particle of mass $4m$ explodes into three fragments. Two of the fragments each of mass m are found to move with a speed is each in mutually perpendicular directions. The total energy released in the process of explosion is _____ mv^2.

Q34. Two particles of mass 1 kg and 3 kg move towards each other under their mutual force of attraction. No other force acts on them. When the relative velocity of approach of two particles is $2\,m/s$, their centre of mass has a velocity of $0.5\ m/s$. When the relative velocity of approach becomes $3m/s$, the velocity of centre of mass is _____

Q35. A particle of mass $10\ kg$ is moving with velocity of $\left(2\hat{i}+3\hat{j}\right)m/s$. From time $t=0$, a force $10\,\hat{j}$ starts acting on the particle. The x- component of momentum of the particle at any other time t is _____.

Q36. Two blocks of masses $10\ kg$ and $4\ kg$ are connected by a spring of negligible mass and placed on a frictionless horizontal surface. An impulse gives a velocity of $14\ m/s$ to the heavier block in the direction of the lighter block. The velocity of centre of mass is _____.

Q37. A $4.0\ kg$ block sliding on a frictionless surface explodes into two fragments of $2.0\ kg$ parts. One moves with $3.0\ m/s$ due north and other at $6.0\ m/s$, 30^0 north east. The original speed of the block is _____.

MSQ (Multiple Select Questions)

Q38. If for a system of N particles of different masses $m_1, m_2 ... m_N$ with position vectors $\vec{r}_1, \vec{r}_2, ... \vec{r}_N$ and corresponding velocities $\vec{v}_1, \vec{v}_2, ... \vec{v}_N$ respectively such that $\sum_i \vec{v}_1 = 0$, then

(a) linear momentum must be constant

(b) total angular momentum must be independent of the choice of the origin

(c) the total force on the system must be zero

(d) total torque on the system must be zero

Q39. A ball bits the floor and rebounds, after elastic collision in this case

(a) the magnitude of the momentum of the ball just after collision is same as that just before collision

(b) the mechanical energy of the ball remains the same in the collision.

(c) total momentum of ball and earth is conserved

(d) total kinetic energy of ball and earth is conserved

Q40. A net force $F_x(t) = A + Bt^2$ in the $+x$ direction is applied to a boy of mass m. The force starts at $t = 0$ and continues until time t

(a) Impulse of the force is $2Bt$ (b) His speed at time t is $\frac{A}{m}t + \frac{B}{3m}t^3$

(c) Impulse of the force is $At + \frac{B}{3}t^3$ (d) Speed at time t is $\frac{2B}{m}t$

Q41. A block of mass m moving on a smooth horizontal plane with a velocity v_0 collides with a stationary block of mass M at the back of which a spring of spring constant k is attached as shown in figure. Select the correct option(s).

(a) Velocity of centre of mass $\frac{m}{m+M}v_0$

(b) Initial kinetic energy of the system in centre of mass frame is $\frac{1}{4}\left(\frac{mM}{m+M}\right)v_0^2$

(c) Maximum compression in the spring is $v_0\sqrt{\frac{mM}{(M+m)}\frac{1}{k}}$

(d) When the spring is in state of maximum compression the kinetic energy in the centre of mass frame is zero.

Q42. A body moving towards a finite body at rest collides with it. It is possible that

(a) both bodies come to rest

(b) both bodies move after collision

(c) the moving body came to rest and stationary body starts moving

(d) the stationary body remain stationary the moving body change its velocity

Q43. Pick out the **CORRECT** alternative(s):

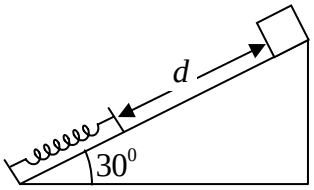
(a) If center of mass of three particles is at rest and it is known that two of them are moving along different lines, then the third must also be moving.

(b) If centre of mass remains at rest, then net work done by the forces acting on the system must be zero.

(c) If centre of mass remains at rest, then the net external force must be zero.

(d) For two particles collision, the linear momentum is conserved but for three particles it is not conserved.

Q44. A block of $4\ kg$ mass starts at rest and slides a distance d down a friction less incline (angle 30^0) when it runs into a spring of negligible mass. The block slides an additional difference of $25\ cm$ before it is brought to rest momentarily by compressing the spring. The force constant of the spring is $400\ N/m$.



(i) The value of d is

(ii) When the block strikes the spring its speed is $\sqrt{3.75}\ m/s$

(iii) At maximum compression the potential energy of the spring is $12.5\ J$

(iv) The change in gravitational P.E. of the block from the top and till the position of momentary rest is $15\ J$

Q45. A body moving towards a body of finite mass at rest collides with it. It is possible that

(a) both bodies come to rest.

(b) both bodies move after collision.

(c) the moving body stops and the body at rest starts moving.

(d) the stationary body remains stationary.

Q46. In an elastic collision between two particles.

(a) the total kinetic energy of the system is always conserved.

(b) the kinetic energy of the system before collision is equal to the kinetic energy of the system after collision.

(c) the linear momentum of the system is always conserved.

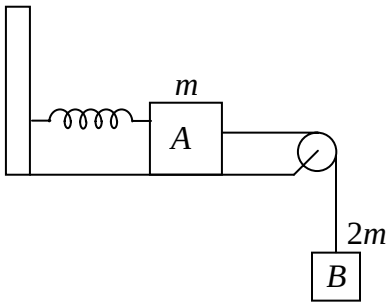
(d) the linear momentum before collision is equal to the linear momentum after collision.

Q47. A particle is taken from point A to point B under the influence of a force field. Now it is taken from B to A and it is observed that the work done in taking the particle from B to A is not equal to the work done in taking it from A to B. If W_{nc} and W_c are the work done by non-conservative and conservative forces and ΔU and ΔK are change in potential and change in kinetic energy then

(a) $W_{nc} - \Delta U = \Delta K$ (b) $W_{nc} = -\Delta U$

(c) $W_{nc} + W_c = \Delta K$ (d) $W_{nc} - \Delta U = -\Delta K$

Q48. In the adjoining figure block A is of mass m and block B is of mass $2\ m$. The spring has force constant K. All the surfaces are smooth and the system is released from rest with spring unstretched.



(a) The maximum extension of the spring is $\frac{4mg}{K}$

(b) The speed of block A when extension in spring is $\frac{2mg}{K}$, is $2g\sqrt{\frac{2m}{3K}}$

(c) The net acceleration of block B when the extension in the spring is maximum is $\frac{2}{3}g$

(d) Tension in the thread for extension of $\frac{2mg}{K}$ in spring is mg.

Q49. At two positions kinetic energy and potential energy of a particle are $K_1 = 10\,J, U_1 = -20\,J, K_2 = 20\,J, U_2 = -10\,J$. In moving from 1 to 2.

(a) Work done by conservative forces is positive

(b) Work done by conservative forces is negative

(c) Work done by all the forces is positive

(d) Work done by all the forces is negative

Q50. The potential energy function of a particle in the x-y plane is given by $U = k(x+y)$ where k is a constant

(a) The work done by the conservative force in moving a particle from $(1,1)$ to $(2,3)$ is $-3k$

(b) The work done by the conservative force in moving the particle from $(0,0)$ to $(1,1)$ is $-2k$

(c) The work done by the conservative force in moving the particle from $(1,1)$ to $(2,2)$ is $+2k$

(d) The work done by the conservative force in moving the particle from $(0,0)$ to $(3,3)$ is $-6k$

Q51. A block is placed on top of an inclined plane inclined at 37^0 with the horizontal. The length of the plane is $5m$. The block slides down the plane and reaches the bottom. Take $g = 10\,m/s^2$, $\sin 37^0 = \frac{3}{5}$ and $\cos 37^0 = \frac{4}{5}$.

(a) If the plane is smooth the block reaches the bottom with a speed of $2\sqrt{15}\,m/s$

(b) If the plane is smooth the block reaches the bottom with a speed of $3\sqrt{15}\ m/s$

(c) If the plane has the coefficient of friction 0.25, the block reaches the bottom with a speed of $2\sqrt{10}\ m/s$

(d) If the plane has coefficient of friction 0.25, the block reaches the bottom with a speed of $5\sqrt{10}\ m/s$

Q52. You lift a suitcase from the floor and keep it on a table. The work done by you on the suitcase does not depend on

(a) the path taken by the suitcase

(b) the time taken by you in doing so

(c) the weight of the suitcase

(d) your weight

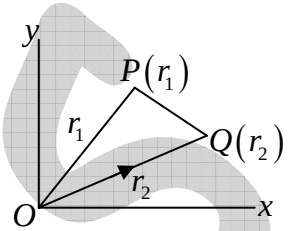
Solutions

MCQ (Multiple Choice Questions)

Ans. 1: (b)

Ans. 2: (d)

Solution: If r_1, r_2 are position vectors of two masses m_1 and m_2 then the position vector of their centre of mass R_{CM} is given as $(m_1+m_2)R_{CM} = m_1r_1 + m_2r_2$



Differentiating w.r.t. time, we get $(m_1+m_2)v_{cm} = m_1v_1 + m_2v_2$

$$(100+20)v_{CM} = 100\left(2\hat{i} - 7\hat{j} + 3\hat{k}\right) + 20\left(-10\hat{i} + 35\hat{j} - 3\hat{k}\right)$$

$$120v_{CM} = 200\hat{i} - 700\hat{j} + 300\hat{k} + \left(-200\hat{i}\right) + 700\hat{j} - 60\hat{k} = 240\hat{k}$$

$$v_{CM} = \frac{240}{120}\hat{k} = 2\hat{k}\ cm/s$$

Ans. 3: (c)

Solution: The displacement is given as $x = \frac{t^2}{2} \Rightarrow dx = \frac{2}{2}t\,dt$ Work done $\Rightarrow dW = F.dx = madx$

(ii) Differentiating equation (i) w.r.t. time, we get

Acceleration $\frac{d^2x}{dt^2} = \frac{2}{2} = 1$ So force $F = ma = 2\times1 \Rightarrow F = 2N$

Work done $dW = 2dx = 2t\,dt \Rightarrow W = \int_0^{t_0} t\,dt = 2.\frac{t_0^2}{2} = t_0^2$

Ans. 4: (a)

Solution: When the collide elastically then their velocities after collision are given as

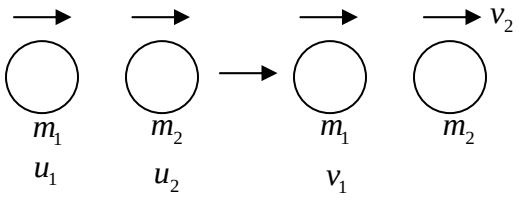
$$v_1 = \frac{(m_1 - m_2)u_1 + 2m_2u_2}{m_1 + m_2}$$

$$v_2 = \frac{2m_1u_1 + (m_2 - m_1)u_2}{m_1 + m_2}$$

Here, $m_1 = m, m_2 = M, u_1 = u, u_2 = 0$

And $v_2 = v$

(i)



Before collision After collision

Putting all these values in equation (i), we get

$$v_1 = \frac{(m-M)u+0}{m+M} \Rightarrow v_1 = \left(\frac{m-M}{m+M}\right)u$$

Ans. 5: (a)

Solution: The velocity of rocket

$$M_0 = \text{ mass at } t=0 \;\; v_0 = \text{velocity at } t=0 \;\; v = v_0 + v_e \log\frac{M_0}{M}, v \propto \log\frac{M_0}{M}$$

Ans. 6: (b)

Solution: Applying conservation of energy

$$= \frac{1}{2}kx^2 + \frac{m^2v^2}{2m} = \frac{1}{2}mv'^2 + 0 \Rightarrow mv'^2 = kx^2 + \frac{m^2v^2}{m}$$

$$\Rightarrow \qquad 2\times v'^2 = 4000\times\left(5\times10^{-2}\right)^2 + \frac{4^2}{2} \Rightarrow 2v'^2 = 0.4\times25+8 \Rightarrow v' = 9 \Rightarrow v' = 3$$

Ans.7: (c)

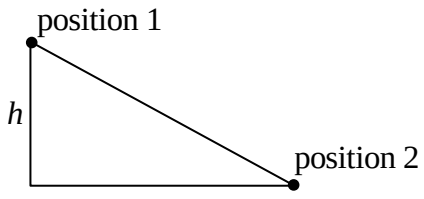
Solution: A force is said to be conservative if the work done by it in a closed loop is zero. Alternatively if the work done by a force is path independent, the force is said to be conservative. Frictional force does not satisfy any of these properties, hence it is a non-conservative force.

Ans.8: (c)

Solution: The concept of potential energy can be associated with only a conservative force. The mechanical energy of a system is the sum of kinetic and potential energy. Internal conservative forces can not change the mechanical energy. The mechanical energy of the system changes by the amount of work done by the external force.

Ans. 9: (d)

Solution: Gravitational potential energy at a height h is mgh, hence $\Delta U = mg\left(h_2 - h_1\right)$. Mechanical energy of position 1



= Mechanical energy of position 2, hence $0 + mgh = \frac{1}{2}mv^2 + 0$

$\therefore v = \sqrt{2gh}$. The change in spring potential energy is $\Delta U = \frac{1}{2}K\left(x_2^2 - x_1^2\right)$

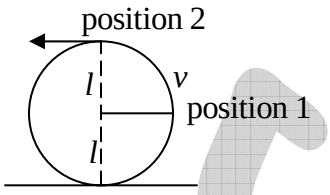
The change in potential energy of a system is equal to the negative of the work done by the conservative force on the system.

Ans. 10: (c)

Solution: The minimum speed at the top for the particle to complete the circle $=\sqrt{gl}$. Now applying the principle of conservation of mechanical energy.

Mechanical energy of position 1 = Mechanical energy of position 2

$$\frac{1}{2}mv^2 + mgl = \frac{1}{2}m\left(\sqrt{gl}\right)^2 + 2mgl$$



or by solving we get $v = \sqrt{3gl}$

Ans. 11: (b)

Solution: $F = k_A x_A = k_B x_B$ or $\frac{x_B}{x_A} = \frac{k_A}{k_B} = 2$

Hence $\frac{E_B}{E_A} = \frac{\frac{1}{2}k_B x_B^2}{\frac{1}{2}k_A x_A^2} = \left(\frac{1}{2}\right)\cdot 2^2 = 2 \quad \therefore E_B = 2E_A = 2E$

Ans. 12: (d)

Solution: Linear momentum of a system is $\vec{P} = M\vec{V}_{cm}$

If $\vec{P}$ is constant we can only conclude that $\vec{V}_{cm}$ is constant, if $\vec{V}_{cm}$ is constant then $\vec{P}$ is constant.

Ans. 13: (d)

Solution: Since the motion of centre of mass is decided by the net external force acting on the system and the net external force acting on the system and the net external force is the same (both before and after breaking), hence centre of mass does not shift at all.

Ans. 14: (c)

Solution: From conservation of momentum $M \times 0 = \frac{M}{2}\vec{v}_1 + \frac{M}{2}\vec{v}_2 \Rightarrow \vec{v}_1 = -\vec{v}_2$

Ans. 15: (b)

Solution: From the relation $\vec{F}_{ext} = M\vec{a}_{cm}$ if $\vec{F}_{ext} = 0$, then $\vec{a}_{cm} = 0$

From the relation $\vec{P} = M\vec{V}_{cm}$, if $\vec{V}_{cm}$ is constant, $\vec{P}$ is constant. During collision the linear momentum is necessarily constant (in the absence of external forces).

Conservation of linear momentum and conservation of energy are entirely different concepts.

Ans. 16: (d)

Solution: From the relation $\vec{F}_{ext} = m\vec{a}_{cm}$ we have $m\vec{g} = m\vec{a}_{cm}$ or $\vec{a}_{cm} = \vec{g}$

Ans. 17: (b)

Solution: From the relation $\vec{F} = -\frac{\partial U}{\partial x}\hat{i} - \frac{\partial U}{\partial y}\hat{j}$

We see that the most general expression is $U = -2x^2y + \text{constant}$.

Ans. 18: (c)

Solution: From the question,

$$\frac{1}{2}k(2\,cm)^2 = 100 \qquad \text{(i)}$$

$$\frac{1}{2}k(4\,cm)^2 = E \qquad \text{(ii)}$$

Hence $\frac{E}{100} = 4 \Rightarrow E = 400\,J$, change in energy $= 400 - 100 = 300\,J$.

Ans. 19: (a)

Solution: Work done by air resistance = decrease in mechanical energy

$$\Rightarrow 0.1mgl = \frac{1}{2}mv^2 - mgl \qquad \therefore v = \sqrt{1.8\,gl} = \sqrt{36} = 6\ m/s$$

Ans. 20: (b)

Solution: $M \cdot 20 = \frac{M}{2}\cdot 30 + \frac{M}{2}v$ or $40 = 30 + v \Rightarrow v = 10\ m/s$

Ans. 21: (b)

Solution: Even after the vertical fall, the horizontal component of momentum is not affected, hence

$$Mv = (M+m)v' \qquad \text{or} \qquad v' = \frac{mv}{M+m}$$

Ans. 22: (c)

Solution: since the suitcase is at rest in both situation, hence

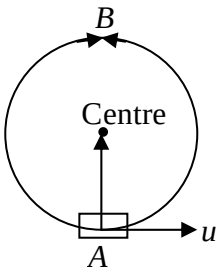
$Wg + W_{you} = 0$ (work KE theory)

$W_{you} = -Wg = -(-mgh) = mgh$. Hence only option (c) is correct

NAT (Numerical Answer Type)

Ans. 23: $10\ m/s$

Solution: If u is velocity at the lowest point A then in order en reach the mass at highest point B the velocity u should be given as $u^2 \geq 5gr$.



$$u = \sqrt{5gr} = \sqrt{5\times 10\times 2} = \sqrt{100} = 10\ m/s$$

Ans. 24: $49\ kg\ s^{-1}$

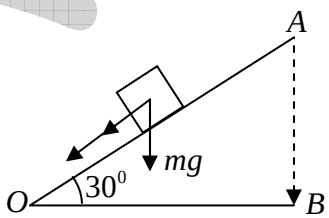
Solution: If $\frac{dm}{dt}$ is the rate exhaust gas then it is related with speed of rocket as

$$v\frac{dm}{dt} = Mg \quad \Rightarrow 2\times 10^3 \frac{dm}{dt} = 10^4 \times 9.8 \quad \Rightarrow \frac{dm}{dt} = \frac{9.8\times 10^4}{2\times 10^3} = 49\ kg\ s^{-1}$$

Ans. 25: 450

Solution: By conservation of energy law.

Total energy at $A =$ Total energy at O



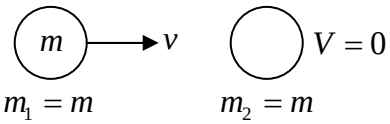
$$\Rightarrow \quad (KE)_A + (PE)_A = (KE)_o + (PE)_O + (\text{Frictional energy})$$

$$\Rightarrow \quad 0 + MgAB = \frac{1}{2}mv^2 + 0 + W_t \Rightarrow Mg\,10\sin 30 = \frac{1}{2}mv^2 + W_t$$

$$\Rightarrow \quad 10\times 25\times 10\times \frac{1}{2} = \frac{1}{2}\times 25\times (8)^2 + W_t \Rightarrow 1250 = 800 + W_t \Rightarrow W_t = 450\ J$$

Ans. 26: 0.25

Solution: Lets v_1 is velocity of first mass after collision and v_2 is velocity of second mats after collision. By the conservation law momentum



$$m_1 v = m_1 v_1 + m_2 v_2$$

But $\quad m_1 = m_2 = m \Rightarrow mv = mv_1 + mv_2 \Rightarrow v = v_1 + v_2$

By conservation law of energy

$$\frac{1}{2}mv^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2 \Rightarrow v^2 = v_1^2 + v_2^2$$

Solving equation (ii) and (iii), we get

Velocity of mass becomes zero whereas velocity of sector mass attain the velocity of first.

Now, the velocity of centre of mass is given

$$(m+m)v_{CM} = mv \Rightarrow v_{CM} = \frac{1}{2}v$$

Hence, the velocity of first w.r.t. centre of mass

$$0 - v_{CM} = -\frac{1}{2}v$$

Total kinetic energy as seen centre of mass

$$\frac{m(v_1 - v_{cm})^2}{2} + \frac{m(v_2 - v_{cm})^2}{2} = \frac{m\left(v_1 - \frac{v}{2}\right)^2}{2} + \frac{m\left(v_1 - \frac{v}{2}\right)^2}{2}$$

using $v^2 = v_1^2 + v_2^2$ and $v = v_1 + v_2$ $= E = \frac{mv^2}{4}$

Ans. 27: -1

Solution: Work done $= \vec{F}.d\vec{r}$

Here, $\vec{F} = \hat{x} + 2\hat{y} + 3\hat{z}\ N$

$\vec{r}_1 = (\hat{x} + \hat{y} + \hat{z}); \vec{r}_2 = (\hat{x} - \hat{y} + 2\hat{z}), dr = (2\hat{y} + \hat{z})$

So, $F = (\hat{x} + 2\hat{y} + 3\hat{z}).(-2\hat{y} + \hat{z}) = 2(-2) + 3(1) = -4 + 3 = -1J$

Ans. 28: 0.15

Solution: Taking block & spring as the system and applying the conservation of mechanical energy

$$\frac{1}{2} \times 0.5 \times (1.5)^2 + 0 = 0 + \frac{1}{2} \times 50 \times x_{\max}^2 \Rightarrow x_{\max} = 0.15\ m$$

Ans. 29: 1

Solution: From conservation of linear momentum $m \times 3 = (m + 2m) \times v$ or $v = \frac{3m}{3m} = 1m/s$

Ans. 30: 2

Solution: Taking the zero of spring potential energy of the location of unstretched spring and the zero of gravitational potential energy at the lowest point.

From conservation of mechanical energy $0 + Mgd = \frac{1}{2}kd^2 + 0$ or $d = \frac{2Mg}{k}$

Ans. 31: 0.25

Solution: The height attained by the projectile

$H = \frac{v_0^2 \sin^2\theta_0}{2g}$, where v_0 is the speed of projection and θ_0 is the initial angle with the horizontal.

$= \frac{v_0^2 \cdot \frac{3}{4}}{2g}$ or $H = \frac{3v_0^2}{8g}$

From conservation of mechanical energy $E + 0 = \frac{mg \cdot 3v_0^2}{8g}$

But $\frac{1}{2}mv^2 = E \quad \Rightarrow E = K + \frac{3}{4}E \Rightarrow K = \frac{1}{4}E$

Ans. 32: 3.6

Solution: Work done in putting the chain = change in gravitational potential energy

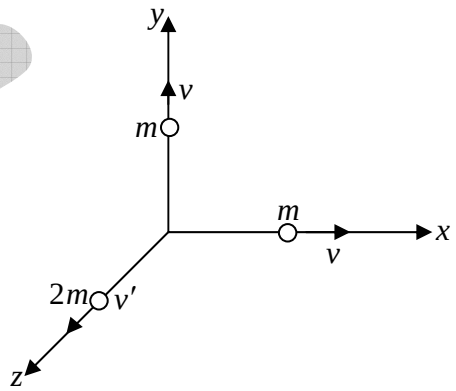
$= 0 - \left[-\left(\frac{4}{2}\right)(0.6) \times 10 \times (0.3) \right] = 3.6\ J$

Ans. 33: 1.5

Solution: From conservation of momentum

$0 = mv\hat{i} + mv\hat{j} + 2m\vec{v}'$ or $\vec{v}' = -\frac{v}{2}\hat{i} - \frac{v}{2}\hat{j}$

$\frac{1}{2}.2mv'^2 = \frac{1}{2}.2m.\left(\frac{v^2}{4} + \frac{v^2}{4}\right) = \frac{1}{2}mv^2$



hence total energy released

$\frac{1}{2}mv^2 + \frac{1}{2}mv^2 + \frac{1}{2}mv^2 = \frac{3}{2}mv^2 = 1.5mv^2$

Ans. 34: $0.5\,m/s$

Solution: When no net external force acts on a system the total linear momentum and hence the velocity of centre of mass remains constant. Hence in both cases velocity of centre of mass $0.5\,m/s$.

Ans. 35: $20\ kg\ ms^{-1}$

Solution: At $t=0$, the x-component of momentum is $10\times 2=20\ kg\ ms^{-1}$

Since no of net external force acts on the particle in the x-direction, hence its x-component of momentum remains constant.

Ans. 36: $10\ m/s$

Solution: The velocity of centre of mass is given by

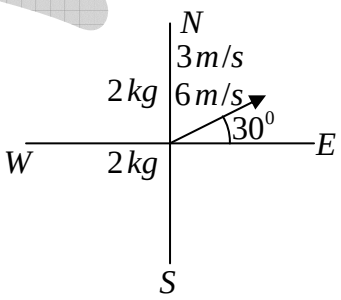
$$V_{cm}=\frac{m_1\vec{v}_1+m_2\vec{v}_2}{m_1+m_2}$$

$$\therefore V_{cm}=\frac{10\times 14+4.0}{10+4}=10\ m/s$$

Ans. 37: 3.43

Solution: From conservation of momentum

$$M\vec{V}=m_1\vec{v}_1+m_2\vec{v}_2$$



or $$4\vec{V}=2\left(6\cos 30\hat{i}+6\sin 30\hat{j}\right)+2\times 3\hat{j}$$

$$4\vec{V}=2\left(\frac{6.\sqrt{5}}{2}\hat{i}+\frac{6}{2}\hat{j}\right)+6\hat{j}=6\sqrt{3}\hat{i}+3\hat{j}+6\hat{j}=6\sqrt{3}\hat{i}+9\hat{j}$$

$$\vec{V}=\frac{6\sqrt{3}}{4}\hat{i}+\frac{9}{4}\hat{j}\Rightarrow V=\frac{1}{4}\sqrt{108+81}=\frac{\sqrt{189}}{4}\ m/s=3.43\,m/s$$

MSQ (Multiple Select Questions)

Ans. 38: (a) and (c)

Solution: $\vec{v}_{cm}=\dfrac{m_1\vec{v}_1+m_2\vec{v}_2+..m_N\vec{v}_N}{m_1+m_2+...+m_N}\Rightarrow\vec{v}_{cm}=\dfrac{\sum_{i=1}^{N}m_i\vec{v}_i}{\sum_{i=1}^{N}m_i}$ $\quad\because\sum\vec{v}_i=0$

$$\Rightarrow\quad\sum\frac{d\vec{v}_i}{dt}=0\Rightarrow\ \frac{1}{m_i}\sum m_i\frac{d\vec{v}_i}{dt}=0\Rightarrow\quad\frac{1}{m_i}\sum dF_i=0\Rightarrow\sum dF_i=0$$

Hence, total force on the system is zero so linear momentum is conserve.

Ans. 39: (c) and (d)

Ans. 40: (b) and (c)

Solution: $F = A + Bt^2$

Impulse, $I = \int_0^t F dt = \int_0^t A + Bt^2 = At + \frac{Bt^3}{3}$

Acceleration, $a = \frac{F}{m} = \frac{A+Bt^2}{m} \Rightarrow \frac{dv}{dt} = \frac{A+Bt^2}{m} \Rightarrow \int_0^v dv = \int_0^t \frac{A+Bt^2}{m} dt \Rightarrow v = \frac{A}{m}t + \frac{B}{3m}t^3$

Ans. 41: (a), (c) and (d)

Solution: Velocity of centre of mass $v_{CM} = \frac{m \times v_0 + M \times 0}{m+M} \Rightarrow v_{CM} = \frac{m\omega_0}{m+M}$

For maximum compression, velocity of each mass become same let it is V

$$\Rightarrow \quad \frac{1}{2}mv_0^2 = \frac{1}{2}(m+M)v^2 \Rightarrow v = \sqrt{\frac{m}{(m+M)}}v_0$$

$$\Rightarrow \quad \frac{1}{2}kx^2 = \frac{1}{2}mv_0^2 - \frac{1}{2}\frac{m^2}{(m+M)}v_0^2 - \frac{1}{2}\frac{mM}{(m+M)}v_0^2$$

on solving, $x = u_0\sqrt{\frac{mM}{M+m}\frac{1}{k}}$

Initial $KE \quad = \frac{1}{2}mv_{cm}^2 + \frac{1}{2}Mv_{cm}^2 = \frac{1}{2} \times \frac{m^2}{(m+M)^2}v_0^2[m+M] = \frac{1}{2}\frac{m^2}{(M+m)}v_0^2$

Ans. 42: (b), (c) and d)

Ans. 43: (a) and (c)

Solution: By definition $\vec{V}_{cm} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3}{m_1+m_2+m_3}$

$\vec{V}_{cm} = 0$ and $\vec{v}_1 \neq 0$ and $\vec{v}_2 \neq 0$ and the lines of $\vec{v}_1$ and $\vec{v}_2$ are different implies $m_1\vec{v}_1 + m_2\vec{v}_2 \neq 0$

hence $\vec{v}_3 = \frac{-(m_1\vec{v}_1 + m_2\vec{v}_2)}{m_3} \neq 0$

If centre of mass is at rest then $\vec{a}_{cm} = 0$ hence $\vec{F}_{\text{exp}} = 0$, but this does not imply net work done by the forces acting on the system to be zero. Linear momentum (in the absence of external force) is conserved for collision of any number of particles.

Ans. 44: (a), (b) and (c)

Solution: Loss of gravitational potential energy = gain in spring potential energy

$$mg\sin\theta\left(d+0.25\right)=\frac{1}{2}kd^2 \Rightarrow 4\times10\times\frac{1}{2}\left(d+0.25\right)=\frac{1}{2}\times400\times\left(0.25\right)^2$$

$$d+0.25=0.375, d=0.375\ m \qquad \text{or} \qquad d=37.5\ cm$$

The speed of the block of the time of striking the spring is

$$mgd\sin30^0=\frac{1}{2}mv^2 \quad \Rightarrow 10\times0.375\times\frac{1}{2}=\frac{1}{2}v^2 \Rightarrow v=\sqrt{3.75}\ m/s$$

At maximum compression potential energy of spring is

$$U=\frac{1}{2}kx^2 \qquad \text{or} \qquad U=\frac{1}{2}\times400\times\left(0.25\right)^2=12.5\ J$$

Ans. 45: (b), (c) and (d)

Solution: Since one body is moving the net linear momentum of the system is non-zero. If both bodies come to rest the net linear momentum will be zero which is not possible.

Yes both bodies may move so as to conserve momentum.

If this is an elastic collision of two equal masses, then the moving body will stop and the body at rest will move.

If the stationary body is of infinite mass, it will remain at rest and the moving body will rebound.

Ans. 46: (b), (c) and (d)

Solution: In elastic collision, the kinetic energy is not conserved during the process of collision, but it is conserved just before and just after the collision.

In any type of collision (in the absence of external forces) the momentum is conserved.

Ans. 47: (a) and (c)

Solution: By definition $W_c = -\Delta U$ also $W_c + W_{nc} = \Delta k$

Hence, $-\Delta U + W_{nc} = \Delta k$

Ans. 48: (a)

Solution: Initially the system is at rest and finally the system is at rest. Applying the principle of conservation of mechanical energy for maximum compression.

Loss in gravitational potential energy = given in spring potential energy

$$\Rightarrow \qquad 2mgx_{\max} = \frac{1}{2}kxx_{\max}^2 \qquad \text{or} \qquad x_{\max} = \frac{4mg}{k}$$

when the extension is $\frac{2mg}{k}$, applying it conservation of mechanical energy gives

$$2m.\frac{2mg}{k}g = \frac{1}{2}mv^2 + \frac{1}{2}.2mv^2 \Rightarrow \frac{4m^2g^2}{k} = \frac{3mv^2}{2} \qquad \text{or } v = \sqrt{\frac{8mg}{3k}} = 2\sqrt{\frac{2mg}{3k}}$$

Ans. 49: (b) and (c)

Solution: $W_C = -\Delta U$, hence $W_C = -(-10+20) = -10\ J$

From work-kinetic energy theorem

Work done by all forces = charge in kinetic energy

$$\Rightarrow \qquad W_{all} = 20 - 10 = 10\ J$$

Ans. 50: (a), (b) and (d)

Solution: Work done by conservation force $= -\Delta U$

$$W_{(1,1)\to(2,3)} = -3k\ ,\ W_{(0,0)\to(1,1)} = -2k\ ,\ W_{(1,1)\to(2,2)} = -2k\ ,\ W_{(0.0)\to(3,3)} = -6k$$

Ans. 51: (a) and (c)

Solution: If the plank is smooth, applying the principle of conservation of mechanical energy

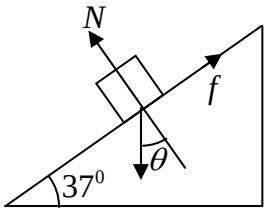
$$5mg\sin 37^0 = \frac{1}{2}mv^2 \quad \text{or} \qquad 5\times 10\times\frac{3}{5} = \frac{1}{2}\times v^2 \Rightarrow v^2 = \sqrt{60} = 2\sqrt{15}\ m/s$$

If the plank has friction, work done by friction force = change in mechanical energy

$$\Rightarrow \qquad \left(-\mu mg\cos 37^0\right)(5) = \left(\frac{1}{2}mv^2 + 0\right) - \left(0 + 5mg\sin 37^0\right)$$

$$\Rightarrow \qquad -5\times(0.25)\times 10\times\frac{4}{5} = \frac{v^2}{2} - 5\times 10\times\frac{3}{5} \Rightarrow -10 = \frac{v^2}{2} - 30$$

$$\frac{v^2}{2} = 20.0 \quad \therefore \quad v^2 = 40 \Rightarrow v = 2\sqrt{10}\ m/s$$



Ans. 52: (a), (b) and (d)

Solution: since the suitcase is at rest in both situation, hence

$$Wg + W_{you} = 0 \text{ (work } KE \text{ theory)}$$

$$W_{you} = -Wg = -\left(-mgh\right) = mgh$$

fiziks

3. Central Force and Kepler's System

3.1 Central Force

In classical mechanics, the **central-force problem** is to determine the motion of a particle under the influence of a single central force. A central force is a force that points from the particle directly towards (or directly away from) a fixed point in space, the center, and whose magnitude only depends on the distance of the object to the center.

In central force potential V is only function of r only a central force is always a conservative force; the magnitude F of a central force can always be expressed as the derivative of a time-independent potential energy

$$\vec{\nabla}\times\vec{F}=\frac{1}{r\sin\theta}\left(\frac{\partial F}{\partial\phi}\right)\hat{\theta}-\frac{1}{r}\left(\frac{\partial F}{\partial\theta}\right)\hat{\phi}=0$$

And the force F is defined as $F=-\frac{\partial V}{\partial r}\hat{r}$ (force is only in radial direction)

3.1.1 Angular Momentum and Areal Velocity

The equation of motion in polar coordinate is given by $m(\ddot{r}-r\dot{\theta}^2)=F_r$ and $m(r\ddot{\theta}+2\dot{r}\dot{\theta})=F_\theta$ but for central force

$$\tau=\vec{r}\times\vec{F}_r \qquad \Rightarrow \tau=r\hat{r}\times-\frac{\partial V}{\partial r}\hat{r}\ ,$$

External torque $\tau=0$ so angular momentum is conserve

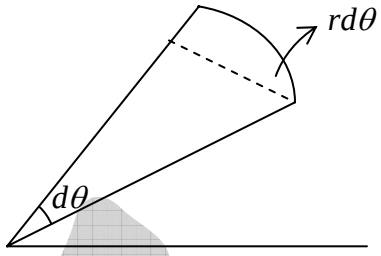
$m(r\ddot{\theta}+2\dot{r}\dot{\theta})=F_\theta$ but for central force $F_\theta=0$ so $m(r\ddot{\theta}+2\dot{r}\dot{\theta})=0\Rightarrow\frac{d(mr^2\dot{\theta})}{dt}=0$ means

Angular momentum of $mr^2\dot{\theta}=J=\left|\vec{r}\times\vec{p}\right| \qquad \Rightarrow\dot{\theta}=\frac{J}{mr^2}$

$\vec{r}\cdot\vec{J}=\vec{r}\cdot(\vec{r}\times\vec{p})=0 \quad\Rightarrow\quad \vec{r}\perp\vec{J}$ hence position vector $\vec{r}$ is perpendicular to angular momentum vector $\vec{J}$ hence $\vec{J}$ is conserve , its magnitude and direction both are fixed so direction of $\vec{r}$ is also fixed.

So motion due to central force is confine into a plane and angular momentum $\vec{J}$ is perpendicular to that plane

Central force problem. Prove that Areal velocity is constant.



For the central force problem. Now $\mathrm{A} = \frac{1}{2} r \cdot r d\theta$

Areal velocity = $\frac{d\mathrm{A}}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} r^2 \dot{\theta}$

$\frac{d\mathrm{A}}{dt} = \frac{1}{2} r^2 \dot{\theta}$ It is given that $\dot{\theta} = \frac{J}{mr^2}$ so $\frac{d\mathrm{A}}{dt} = \frac{J}{2m}$

Which means equal area will swept in equal time

3.1.2 Total Energy of the System

Hence total energy is not explicitly function of time t so $\frac{\partial E}{\partial t} = 0$ one can conclude that total energy in central potential is constant.

$$E = \frac{1}{2} m v^2 + V(r) \text{ and Velocity } \vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

So total energy $E = \frac{1}{2} m\left(\dot{r}^2 + r^2 \dot{\theta}^2\right) + V(r)$

$$= \frac{1}{2} m\dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 + V(r) \text{ it is known } \dot{\theta} = \frac{J}{mr^2}$$

$$= \frac{1}{2} m\dot{r}^2 + \frac{J^2}{2mr^2} + V(r)\ r > 0$$

$\Rightarrow E = \frac{1}{2} m\dot{r}^2 + V_{eff}$ Where $\frac{J^2}{2mr^2} + V(r)$ is identified as effective potential $V_{effective}$

3.1.3 Condition for Circular Orbit

From equation of motion in radial part $m(\ddot{r}-r\dot{\theta}^2)=f(r)$

For circular orbit or radius r_0, $r=r_0$ and $\ddot{r}=0$ $-r\dot{\theta}^2=f(r)\Rightarrow -\frac{J^2}{mr^3}=f(r)$ at $r=r_0$

Which can be also derived by $\left.\frac{\partial V_{effective}}{\partial r}\right|_{r=r_0}=0$ and $\dot{\theta}=\omega_0$ is identified as angular frequency in circular orbit.

radius $r=r_0$ of circular orbit is also identified as stable equilibrium point so $\left.\frac{\partial^2 V_{effective}}{\partial r^2}\right|_{r=r_0}\geq 0$ .if some how particle of mass m is change its orbit without changing is angular momentum and new orbit is bounded then new orbit is identified as elliptical orbit . the angular frequency in new elliptical orbit is $\omega=\sqrt{\frac{\left.\frac{\partial^2 V_{effective}}{\partial r^2}\right|_{r=r_0}}{m}}$

3.1.4 Equation of Motion and Differential Equation of Orbit

From equation of motion in radial part $m(\ddot{r}-r\dot{\theta}^2)=f(r)\Rightarrow \frac{md^2r}{dt^2}-\frac{J^2}{mr^3}=f(r)$ .....(1)

Where $J=mr^2\dot{\theta}\Rightarrow d\theta=\frac{J}{mr^2}dt\Rightarrow \frac{d}{dt}=\frac{J}{mr^2}\frac{d}{d\theta}$

$$\frac{d^2}{dt^2}=\left(\frac{d}{dt}\right)\left(\frac{d}{dt}\right)=\left(\frac{J}{mr^2}\right)\frac{d}{d\theta}\left(\frac{J}{mr^2}\right)\frac{d}{d\theta}$$

Substitute in (1)

$$\frac{J^2}{m}\frac{1}{r^2}\frac{d}{d\theta}\left(\frac{1}{r^2}\frac{dr}{d\theta}\right)-\frac{J^2}{mr^3}=f(r)\Rightarrow \frac{J^2}{m}\frac{1}{r^2}\frac{d}{d\theta}\left(\frac{d(-1/r)}{d\theta}\right)-\frac{J^2}{mr^3}=f(r)$$

$$-\left(\frac{J^2}{m}\frac{1}{r^2}\frac{d^2(1/r)}{d\theta^2}+\frac{J^2}{mr^3}\right)=f(r)\Rightarrow -\frac{J^2}{mr^2}\left(\frac{d^2(1/r)}{d\theta^2}+\frac{1}{r}\right)=f(r)$$

Let $\frac{1}{r}=u\Rightarrow -\frac{J^2u^2}{m}\left(\frac{d^2u}{d\theta^2}+u\right)=f\left(\frac{1}{u}\right)$ (differential equation of an orbit)

Example: Consider that the motion of a particle of mass m in the potential field $V(r)=\frac{kr^2}{2}$ If l is angular momentum.

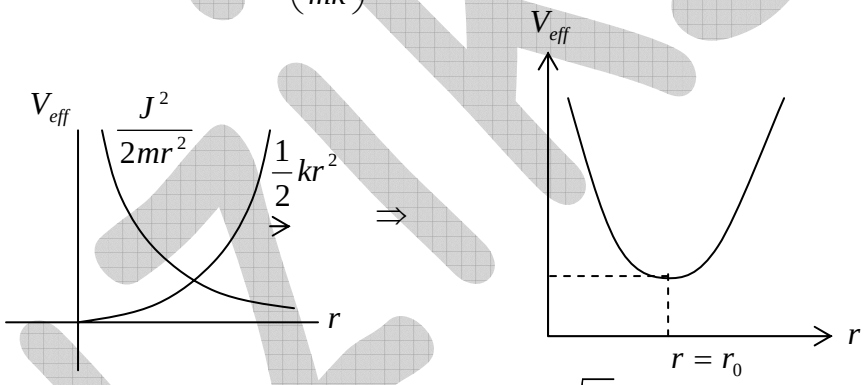
(a) What is effective potential (V_{eff}) of the system. plot V_{eff} vs r

(b) Find value of energy such that motion is circular in nature.

(c) If particle is slightly disturbed from circular orbit such that its angular remain constant. What will nature of new orbit? Find the angular frequency of new orbit in term of m, l, k.

Solution: (a) $V_{eff}=\frac{J^2}{2mr^2}+\frac{1}{2}kr^2$

(b) $\frac{dV_{eff}}{dr}=-\frac{J^2}{mr^3}+kr=0$ at $r=r_0$ so $r_0=\left(\frac{J^2}{mk}\right)^{1/4}$ and $J=m\omega_0 r_0^2$



for circular motion $m\omega_0^2 r_0 = kr_0$ where r_0 is radius of circle $\omega_0=\sqrt{\frac{k}{m}}$

total energy $E=\frac{J^2}{2mr^2}+\frac{1}{2}kr^2=\frac{mkr_0^4}{2mr_0^2}+\frac{1}{2}kr_0^2$ $E=kr_0^2$ put $r_0=\left(\frac{J^2}{mk}\right)^{1/4}$ $E=J\sqrt{\frac{k}{m}}$

(c) orbit is elliptical in nature

$$\left.\frac{d^2V_{eff}}{dr^2}\right|_{r=r_0}=\frac{3J^2}{mr^4}+k=\frac{3J^2}{m\left(\frac{J^2}{mk}\right)}+k=4k$$

$$\omega=\sqrt{\frac{\left.\frac{d^2V_{eff}}{dr^2}\right|_{r=r_0}}{m}}\Rightarrow\omega=\sqrt{\frac{4k}{m}}\Rightarrow\omega=2\sqrt{\frac{k}{m}}\Rightarrow 2\omega_0$$

Example: A particle of mass m moves under the influence of an attractive central force $f(r)$.

(a) What is condition that orbit is circular in nature if J is the angular momentum of particle

(b) If force is in form of $f(r)=\frac{-k}{r^n}$ determine the maximum value of n for which the circular orbit can be stable.

Solution: (a) if $V_{eff}=\frac{J^2}{2mr^2}+V(r)$ for circular stable orbit $\frac{\partial V_{eff}}{\partial r}=0,\ \frac{\partial^2 V_{eff}}{\partial r^2}>0$

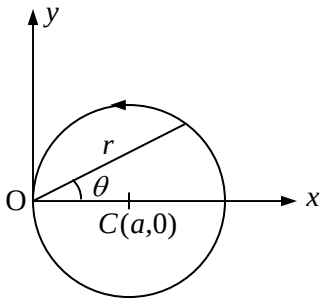
(b) $f(r)=\frac{-k}{r^n}$ for circular motion $\frac{\partial V_{eff}}{\partial r}=0\Rightarrow -\frac{J^2}{mr^3}+\frac{\partial V}{\partial r}=0$

It is given $\frac{\partial V}{\partial r}=-f(r)$ if $f(r)=\frac{-k}{r^n}\Rightarrow\frac{\partial V}{\partial r}=\frac{k}{r^n}$

$$-\frac{J^2}{mr^3}+\frac{k}{r^n}=0\Rightarrow\frac{k}{r^n}=\frac{J^2}{mr^3}$$

$$\frac{\partial^2 V_{eff}}{\partial r^2}>0\Rightarrow\frac{3J^2}{mr^4}-\frac{nk}{r^{n+1}}>0\Rightarrow\frac{3J^2}{mr^4}-\frac{n}{r}.\frac{J^2}{mr^3}>0\ \text{so}\ n<3$$

Example: A particle of mass *m* and angular momentum *l* is moving under the action of a central force *f*(r) along a circular path of radius *a* as shown in the figure. The force centre *O* lies on the orbit.



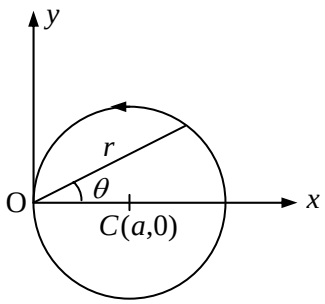
(a) Given the orbit equation in a central field motion.

$$\frac{d^2u}{d\theta^2}+u=-\frac{m}{l^2u^2}f\ ,\ \text{where}\ u=\frac{1}{r}.$$

Determine the form of force in terms of l, m, a *and* r.

(b) Calculate the total energy of the particle assuming that the potential energy $V(r)\to 0$ as $r\to\infty$.

Solution: (a) from the fig $r = 2a\cos\theta$ $\quad \frac{1}{r} = \frac{\sec\theta}{2a}$



$$-\frac{J^2u^2}{m}\left[\frac{d^2u}{d\theta^2}+u\right] = f\left(\frac{1}{u}\right)$$

$$-\frac{J^2\sec^2\theta}{2am}\left[\frac{1}{2a}\left(\sec\theta\tan^2\theta+\sec^3\theta\right)+\frac{\sec\theta}{2a}\right] = f\left(\frac{1}{u}\right)$$

$$-\frac{J^2\sec^2\theta}{2am}\left[\frac{1}{2a}\left(\sec\theta\tan^2\theta+\sec^3\theta+\sec\theta\right)\right] = f\left(\frac{1}{u}\right)$$

$$-\frac{J^2\sec^3\theta}{4a^2m}\left[\tan^2\theta+\sec^2\theta+1\right] = f\left(\frac{1}{u}\right) \Rightarrow -\frac{2J^2\sec^5\theta}{4a^2m} = f\left(\frac{1}{u}\right) \Rightarrow f(r) \propto \frac{1}{r^5}$$

(b) $E = \frac{m\dot{r}^2}{2}+\frac{J^2}{2mr^2}+V(r)$, $r\to\infty$, $V(r)\to 0$ $\quad \frac{J^2}{2mr^2}\to 0 \quad r\to\infty$

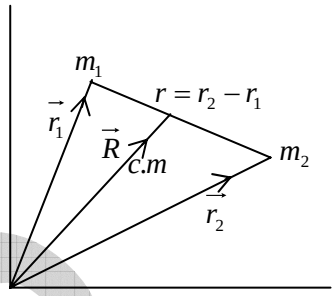
$$E = \frac{m\dot{r}^2}{2} \text{ and } r = 2a\cos\theta \text{ and } \dot{r} = -2a\sin\theta\dot{\theta}, \quad \dot{\theta} = \frac{J^2}{mr^2} \text{ as } r\to\infty$$

hence $\dot{\theta} = \frac{J^2}{mr^2}\to 0$ so $\dot{r}\to 0$ so $E = 0$

3.2 Two Body Problem

Reduction of two body central force problem to the equivalent one body problem

A system of two particles of mass m_1 and m_2 whose instantaneous position vectors of inertial frame with origin O are r_1 and r_2 respectively.



Vector m_2 relative to m_1 is $\vec{r} = \vec{r_2} - \vec{r_1}$

The potential energy is V is only function for distance between the particles

So $V = V(|\vec{r}_2 - \vec{r}_1|)$

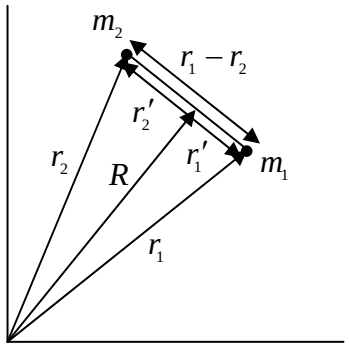
Total energy of the system is given by in lab frame

$$E = \frac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec{r}}_2^{\,2} + V(r_2 - r_1)$$

Let the position vectors of m_1 and m_2 be r_1 and r_2. The position vector of the center of mass, measured from the same origin, is

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$$

The center of mass lies on the line joining m_1 and m_2. To show this, suppose first that the tip of R does not lie on the line, and consider the vectors r_1', r_2' from the tip of R to m_1 and m_2. From the sketch we see that



$$\vec{r}_1' = \vec{r}_1 - \vec{R}$$

$$\vec{r}_2' = \vec{r}_2 - \vec{R}$$

$$\vec{r}_1' = \vec{r}_1 - \frac{m_1 \vec{r}_1}{m_1 + m_2} - \frac{m_2 \vec{r}_2}{m_1 + m_2} = \frac{m_2}{m_1 + m_2}(\vec{r}_1 - \vec{r}_2)$$

$$\vec{r}_2' = \vec{r}_2 - \frac{m_1 \vec{r}_1}{m_1 + m_2} - \frac{m_2 \vec{r}_2}{m_1 + m_2} = -\left(\frac{m_1}{m_1 + m_2}\right)(\vec{r}_1 - \vec{r}_2)$$

r_1' and r_2' are proportional to $r_1 - r_2$, the vector from m_1 to m_2. Hence r_1' and r_2' lie along the line joining m_1 and m_2 as shown. Furthermore,

$$r_1' = \frac{m_2}{m_1 + m_2}|r_1 - r_2| = \frac{m_2}{m_1 + m_2} r \text{ and } r_2' = \frac{m_1}{m_1 + m_2}|r_1 - r_2| = \frac{m_2}{m_1 + m_2} r$$

$E=\frac{1}{2}m_1\left|\vec{r}_1'\right|^2+\frac{1}{2}m_2\left|\vec{r}_2'\right|^2+V\left(r_2-r_1\right)$. The total energy is transformed

$$E=\frac{1}{2}\left(m_1+m_2\right)\left|\dot{\vec{R}}\right|^2+\frac{1}{2}\frac{m_1m_2}{m_1m_2}\left|\dot{\vec{r}}\right|^2-V\left(r\right)$$

Centre of mass moving with constant momentum and equation of motion for three generalized co-ordinates or will not terms in R and $\dot{R}$. Hence discuss the motion of system one can ignore $\frac{1}{2}\left(m_1+m_2\right)\dot{R}^2$.

So Energy in centre of mass reference frame reduce to $E=\frac{1}{2}\frac{m_1m_2}{m_1+m_2}\left|\dot{\vec{r}}\right|^2+V\left(r\right)$

Where $\mu=\frac{m_1m_2}{m_1+m_2}$ is reduce to one body system in centre of mass reference frame.

3.2.1 Kepler's Problem

Kepler discuss the orbital motion of the sun and Earth system under the potential $V(r)=-\frac{k}{r}$ where $k=Gm_sm_e$ it is given m_s and m_e is mass of Sun and Earth .Although kepler discuss sun and earth system but method can be used for any system which is interacting with potential $V(r)=-\frac{k}{r}$

The reduce mass for sun and earth system is $\mu=\frac{m_em_s}{m_e+m_s}\Rightarrow\frac{m_e}{1+\frac{m_e}{m_s}}=m_e\ ,m_s>>m_e$

Let us assume mass of earth $m_e=m$

3.2.2 Kepler's First Law: Every planet (earth) moves in an elliptical orbit around the sun, the sun is being at one of the foci. Where sun and earth interact each other with potential $V(r)=-\frac{k}{r}$ we solve equation of motion in center of mass reference frame with reduce mass $\mu=m_e=m$

3.2.3 Equation of Motion:

$$m\ddot{r} - mr\dot{\theta}^2 = -\frac{k}{r^2} \qquad \text{put } \dot{\theta} = \frac{J}{mr^2}$$

$$m\ddot{r} - \frac{l^2}{mr^3} = -\frac{k}{r^2}$$

Equation or orbit is given by $\frac{J^2u^2}{m}\left[\frac{d^2u}{d\theta^2} + u\right] = -f\left(\frac{1}{u}\right)$

$$f(r) = -\frac{k}{r^2} \qquad f\left(\frac{1}{u}\right) = -ku^2 \Rightarrow \frac{J^2u^2}{m}\left[\frac{d^2u}{d\theta^2} + u\right] = +ku^2 \Rightarrow \frac{d^2u}{d\theta^2} + u = \frac{ku^2m}{J^2u^2}$$

$$\frac{d^2u}{d\theta^2} + \left(u - \frac{km}{J^2}\right) = 0 \text{ put } u - \frac{km}{J^2} = y \text{ so } \frac{d^2u}{d\theta^2} = \frac{d^2y}{d\theta^2}$$

The equation reduce to $\frac{d^2y}{d\theta^2} + y = 0$

The solution of equation reduce to $y = A\cos\theta$ $u - \frac{km}{J^2} = A\cos\theta \qquad \Rightarrow \qquad u = \frac{km}{J^2} + A\cos\theta$

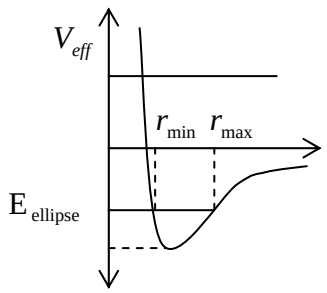
$$\frac{1}{r} = \frac{km}{J^2} + A\cos\theta \Rightarrow \frac{\frac{J^2}{km}}{r} = 1 + \left(\frac{AJ^2}{km}\right)\cos\theta$$

Put $\frac{J^2}{km} = l$ and $e = \frac{AJ^2}{km}$ the equation reduce to $\frac{l}{r} = 1 + e\cos\theta$ which is equation of conics where l is latus rectum and e is eccentricity.

In a central force potential which is interacting with potential $V(r) = -\frac{k}{r}$ can be any conics section depending on eccentricity e.

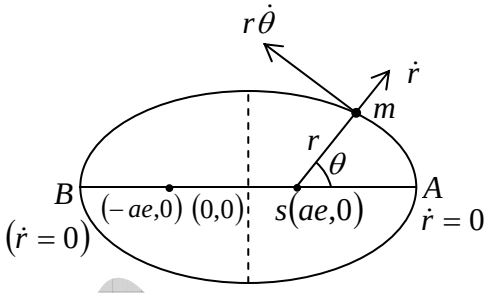
Now we are discuss the case specially of elliptical orbit as Kepler discuss for planetary motion .

Total energy $E = \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} - \frac{k}{r}$ where $V_{effective} = \frac{J^2}{2mr^2} - \frac{k}{r}$ with constant angular momentum J.



If one will plot $V_{effective}$ Vs r it is clear that for negative energy the orbit is elliptical which is shown in figure.

Earth is orbiting in elliptical path with sun as focus as shown in figure.



Let equation of this ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ where $b=a\sqrt{1-e^2}$

minimum value of r is $(a-ae)$ value and maximum value of r is $(a+ae)$ $r_{\max}+r_{\min}=2a$ from plot of effective potential it is identified $r_{\max}$ and $r_{\min}$ is the turning point so so at these points radial velocity is zero

$E=\frac{J^2}{2mr^2}-\frac{k}{r}\Rightarrow 2mEr^2+2mkr-J^2=0$ given equation is quadratic in term of r for their root at $r_{\max}$ and $r_{\min}$. Using theory of quadratic equation sum of root $r_{\max}+r_{\min}=-\frac{2mk}{2mE}$

$\Rightarrow E=-\frac{k}{2a}$ which is negative .

3.2.4 Relationship between Energy and Eccentricity.

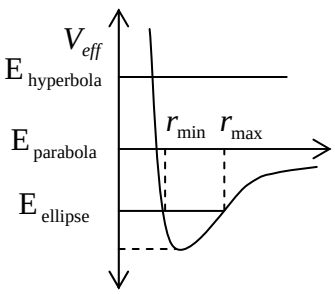
For central potential $V(r)=-\frac{k}{r}$ the solution of orbit is $\frac{l}{r}=1+e\cos\theta$ with $l=\frac{J^2}{km}$

The energy is given by $E=\frac{1}{2}m\dot{r}^2+\frac{J^2}{2mr^2}-\frac{k}{r}$

the solution of orbit is $\frac{l}{r}=1+e\cos\theta$ with $l=\frac{J^2}{km}$

so $\frac{-l}{r^2}\dot{r}=-e\sin\theta\dot{\theta}$ where $\dot{\theta}=\frac{J}{mr^2}$ so $\dot{r}=\frac{eJ\sin\theta}{ml}$,

after putting the value of $\frac{l}{r}=1+e\cos\theta$ and $\dot{r}=\frac{eJ\sin\theta}{ml}$ with $l=\frac{J^2}{km}$ in equation of energy,



one will get $e=\sqrt{1+\frac{2EJ^2}{mk^2}}$

the condition on energy for possible nature of orbit for potential

$E>0$ $e>1$ Hyperbola

$E=0$ $e=1$ Parabola

$E<0$ $e<1$ Ellipse

$E=-\frac{mk^2}{2J^2}$ $e=0$ circle

3.2.5 Kepler's Second Law:

Equal Area will swept in equal time or Areal velocity is constant .

$\frac{dA}{dt}=\frac{J}{2m}$ (which is derived earlier)

3.2.6 Kepler's Third Law:

The square of time period(T) of revolution in elliptical orbit is proportional to cube of semi major axis a ie $T^2 \propto a^3$

$$\frac{dA}{dt}=\frac{J}{2m}\Rightarrow\int dA=\frac{J}{2m}\int dt \Rightarrow \pi ab=\frac{J}{2m}T$$ (πab is the area of ellipse)

$$\pi a\cdot a\sqrt{1-e^2}=\frac{J}{2m}T$$ it is given $e=\sqrt{1+\frac{2EJ^2}{mk^2}}$ and $E=-\frac{k}{2a}$

$$e^2=1-\frac{2kJ^2}{2amk^2}\Rightarrow 1-e^2=\frac{2kJ^2}{2amk^2}$$

$$T^2=\frac{4m^2}{J^2}\pi^2a^2\cdot a^2(1-e^2)$$ put value of $1-e^2=\frac{2kJ^2}{2amk^2}=\frac{J^2}{amk}$

$$T^2=\frac{4m^2}{J^2}\pi^2a^4(1-e^2)\Rightarrow T^2=\frac{4m^2}{J^2}\pi^2a^4.\frac{J^2}{mak}=\frac{4\pi^2ma^3}{k}$$

$$\Rightarrow T^2=\frac{4\pi^2ma^3}{k}$$ if $k=Gm_sm$ then $\Rightarrow T^2=\frac{4\pi^2a^3}{Gm_s}$ where m_s is mass of the sun.

Example: Given a classical model of tritium atom with atom with nucleus of charge +1 and a single.

Electron in a circular orbit of radius r_0 suddenly the nucleus emits a negatron and changes to charge +2 (the emitted negatron escapes rapidly and we can forget about it.) the orbit suddenly has a new situation.

(a) Find the ratio of the electron's energy after to before the emission of the negatron

(b) Describe qualitatively the new orbit

(c) Find the distance of closest and the farthest approach for the new orbits in units of r_0

Solution: (a) As the negatron leaves the system rapidly, we can assume that its leaving has no effect on the position and kinetic energy of the orbiting electron.

From the force relation for the electron,

$$\frac{mv_0^2}{r_0} = \frac{e^2}{4\pi\varepsilon_0 r_0^2}, \text{ and we find its kinetic energy } \frac{mv_0^2}{2} = \frac{e^2}{8\pi\varepsilon_0 r_0}$$

and its total mechanical energy

$$E_1 = \frac{mv_0^2}{2} - \frac{e^2}{4\pi\varepsilon_0 r_0} = -\frac{e^2}{8\pi\varepsilon_0 r_0}$$

before the emission of the negatron. After the emission the kinetic energy of the electron is still $\frac{e^2}{8\pi\varepsilon_0 r_0}$, while its potential energy suddenly changes to $\frac{-2e^2}{4\pi\varepsilon_0 r_0} = \frac{-e^2}{2\pi\varepsilon_0 r_0}$

Thus after the emission the total mechanical energy of the orbiting electron is

$$E_2 = \frac{mv_0^2}{2} - \frac{2e^2}{4\pi\varepsilon_0 r_0} = \frac{-3e^2}{8\pi\varepsilon_0 r_0}, \text{ giving } \frac{E_2}{E_1} = 3.$$

In other words, the total energy of the orbiting electron after the emission is three times as large as that before the emission.

(b) As $E_2 = \frac{-3e^2}{8\pi\varepsilon_0 r_0}$, the condition equation (i) for circular motion is no longer satisfied and the new orbit is an ellipse.

(c) Conservation of energy gives

$$\frac{-3e^2}{8\pi\varepsilon_0 r_0} = \frac{-e^2}{2\pi\varepsilon_0 r_0} + \frac{m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)}{2}$$

At positions where the orbiting electron is at the distance of closest or farthest approach to the atom, we have $\dot{r} = 0$, for which

$$\frac{-3e^2}{8\pi\varepsilon_0 r_0} = \frac{mr^2\dot{\theta}^2}{2} - \frac{e^2}{2\pi\varepsilon_0 r} = \frac{J^2}{2mr^2} - \frac{e^2}{2\pi\varepsilon_0 r}$$

Then with

$$J^2 = m^2 v_0^2 r_0^2 = \frac{me^2 r_0}{4\pi\varepsilon_0}$$

the above becomes

$$3r^2 - 4r_0 r + r_0^2 = 0$$

with solutions $r = \frac{r_0}{3}, \qquad r = r_0$

Hence the distances of closest and farthest approach in the new orbit are respectively

$$r_{\min} = \frac{1}{3}, \qquad r_{\max} = 1$$

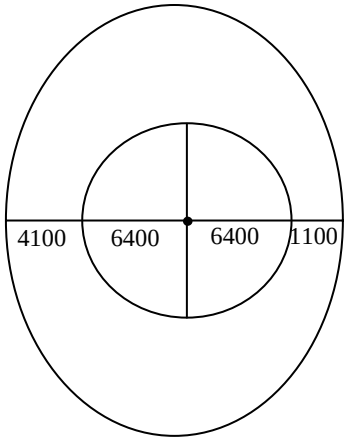
Example: A satellite of mass $m = 2000$ kg is in elliptical orbit about earth. At perigee it has an altitude of 1,100 km and at apogee it has altitude 4,100 km. assume radius of the earth is $R_e = 6,400$ km. it is given $GmM_e = 8\times10^{17}\, J.m$

(a) What is major axis of the orbit?

(b) What is eccentricity of the orbit?

(c) What is angular momentum of the satellite?

(d) How much energy is needed to fix satellite in to orbit from surface of earth?



Solution: $r_{\max} = 4100 + 6400 = 10500\,km$

$r_{\min} = 1100 + 6400 = 7500\,km$

(a) $r_{\max} + r_{\min} = 2a \Rightarrow 18000 = 2a \Rightarrow a = 9000\ km$

$$e = \frac{r_{\max} - r_{\min}}{r_{\max} + r_{\min}}$$

(b) $\Rightarrow e = \dfrac{10500 - 7500}{10500 + 7500} = \dfrac{3000}{18000} = \dfrac{1}{6} \Rightarrow \qquad e = \dfrac{1}{6}$

It is given $k = 8 \times 10^{17}\ J \cdot m$

(c) $E = -\dfrac{k}{2a} = -\dfrac{8 \times 10^{17}}{18000 \times 10^3}$, $E_f = -\dfrac{8}{18} \times 10^{11}\ J = -4.5 \times 10^{10}\ J$

$$e = \sqrt{1 + \frac{2EJ^2}{mk^2}} \Rightarrow \left(\frac{1}{6}\right)^2 = 1 + \frac{2EJ^2}{mk^2}$$

$$\frac{1}{36} - 1 = \frac{2EJ^2}{mk^2} \Rightarrow 140 \times 10^{26} = J^2 \Rightarrow J = \sqrt{140} \times 10^{13} = 1.2 \times 10^{14}\ kgm/\sec^2$$

(d) When satellite is at surface of earth,

$R = 6400\ Km$

$$E_i = -\frac{GMm}{R} = \frac{-8 \times 10^{17}}{6400 \times 10^3} = \frac{-10^{17}}{800 \times 10^3} = \frac{-10^{12}}{8} = -12.5 \times 10^{10}$$

$$E_f = -\frac{GMm}{2a} = -4.5 \times 10^{10}\ J \Rightarrow \Delta E = E_f - E_i = 8 \times 10^{10}\ J$$

Example: For circular and parabolic orbits in an attractive $1/r$ potential having the same angular momentum, show that perihelion distance of the parabola is one-half the radius of the circle.

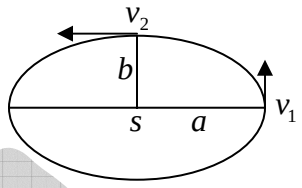
Solution: For Kepler's problem $\dfrac{l}{r} = 1 + \cos\theta$, for circular orbit $e = 0 \Rightarrow \dfrac{l}{r_c} = 1$ and for parabola

$e = 1$ $\dfrac{l}{r} = 1 + \cos\theta$ $\qquad r_p$ is minimum when $\cos\theta$ is maximum.

$\dfrac{l}{r_p} = 2$ and $\dfrac{l}{l_c} = 1 \Rightarrow \dfrac{r_p}{r_c} = \dfrac{1}{2}$

Example: A planet of mass m moves in the inverse square central force field of the Sun of mass M. If the semi-major and semi-minor axes of the orbit are a and b, respectively, the find total energy of the planet by assuming sun is at center of ellipse .

Solution: Assume Sun is at the centre of elliptical orbit.



Conservation of energy $\frac{1}{2}mv_1^2 - \frac{GMm}{a} = \frac{1}{2}mv_2^2 - \frac{GMm}{b}$

Conservation of momentum $L = mv_1 a = mv_2 b$

$$v_2 = v_1\left(\frac{a}{b}\right)$$

$$\frac{1}{2}mv_1^2 - \frac{1}{2}mv_2^2 = \frac{GMm}{a} - \frac{GMm}{b} \Rightarrow \frac{1}{2}m\left(v_1^2 - v_1^2\frac{a^2}{b^2}\right) = GMm\left(\frac{b-a}{ab}\right)$$

$$\frac{1}{2}mv_1^2\left(\frac{b^2-a^2}{b^2}\right) = GMm\left(\frac{b-a}{ab}\right) \Rightarrow \frac{1}{2}mv_1^2 = GMm\left(\frac{b}{a}\right)\cdot\frac{1}{(b+a)}$$

$$E = \frac{1}{2}mv_1^2 - \frac{GMm}{a} = GMm\frac{b}{a}\frac{1}{(b+a)} - \frac{GMm}{a}$$

$$= \frac{GMm}{a}\left(\frac{b}{(b+a)} - 1\right) = \frac{GMm}{a}\left(\frac{b-b-a}{(b+a)}\right) = -\frac{GMm}{(b+a)}$$

MCQ (Multiple Choice Questions)

Q1. A planet moves round the sun. At a point P, it is closest to the sun at distance r_1 and has speed v_1 At another point Q, when it is farthest from the sun at a distance r_2, what is its speed ?

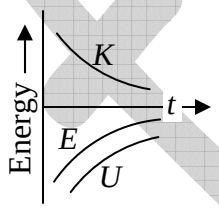
(a) $\frac{r_1^2 v_1}{r_2^2}$ (b) $\frac{r_1 v_1}{r_2}$ (c) $\frac{r_2^2 v_2}{r_1^1}$ (d) $\frac{r_2 v_1}{r_1}$

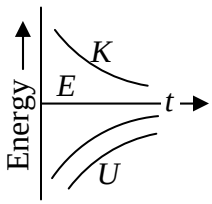
Q2. Two masses m and M are initially at rest at infinite distance apart They approach each other due to gravitational interaction What is their speed of approach at the instant when they are at a distance d apart?

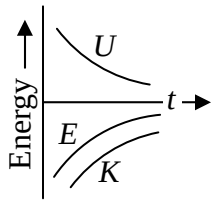
(a) $\left(\frac{2G\left(M^2+m^2\right)^{\frac{1}{2}}}{d}\right)^{\frac{1}{2}}$ (b) $\left(\frac{2GMm}{d\left(M+m\right)}\right)^{\frac{1}{2}}$

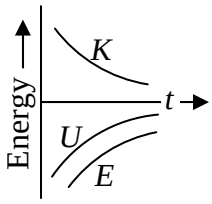
(c) $\left(\frac{2G\left(M+m\right)}{d}\right)^{\frac{1}{2}}$ (d) $\left(\frac{GMm}{d\left(M+m\right)}\right)^{\frac{1}{2}}$

Q3. Which one of t1 following diagrams correct depicts the variation of kinetic energy $\left(K\right)$ potential energy $\left(U\right)$ and total energy $\left(E\right)$ of a body in circular planetary motion? (r is the radius of the circle).

(a)


(b)


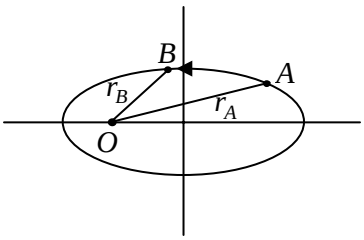
(c)


(d)


Q4. A planet of mass m moves in a circular orbit of radius r_0 in the gravitational potential $V(r) = -\frac{k}{r}$, where k is a positive constant. The orbit angular momentum of the planet is

(a) $2r_0 km$ (b) $\sqrt{2r_0 km}$ (c) $r_0 km$ (d) $\sqrt{r_0 km}$

Q5. The figure given shows a satellite of mass m orbiting elliptically about the centre O of the Earth. The mass of the Earth is M. The work done on the satellite as it moves from A to B in its orbit is

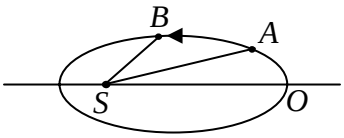


(a) $\frac{GMm}{r_B}\left(1-\frac{r_B}{r_A}\right)$ (b) $\frac{GMm}{r_A}\left(1-\frac{r_A}{r_B}\right)$ (c) $GMM\left(1-\frac{r_B}{r_A}\right)$ (d) $\frac{GMm}{r_B^2 - r_A^2}$

Q6. If the gravitational force is assumed to vary inversely as the nth power of distance r, then how does the time period of a, planet the sun depend upon r?

(a) r^n (b) r^{-n} (c) $r^{(n+1)/2}$ (d) $r^{(n-1)/2}$

Q7. The figure below shows the motion of a planet around the sun in an elliptical orbit with sun as the focus. Areas SOA and SAB shown in the figure can be assumed to be equal. If t_1 and t_2 represent the times for the planet to move from O to A and A to B respectively; then



(a) $t_1 < t_2$ (b) $t_1 > t_2$ (c) $t_1 = t_2$ (d) None of these

Q8. If R is the radius of the earth ρ is mean density and G the gravitational constant, then the earth's surface potential will be nearly equal to

(a) $\frac{\pi\rho G}{R^2}$ (b) $\frac{4}{3}\pi R^3 \rho G$ (c) $\frac{4}{3}\pi\rho R^2 G$ (d) $\frac{4}{3}\pi\frac{R}{\rho}G$

Q9. Three objects S_1, S_2 and S_3 having same mass m are in different elliptic orbits of same semi-major axis, about an object of mass M. If the eccentricities of the orbits are e_1, e_2 and e_3 respectively such that $e_1 < e_2 < e_3$ and if E_1, E_2 and E_3 are their respective mechanical energies, then which one of the following is correct?

(a) $E_1 < E_2 < E_3$ (b) $E_1 > E_2 > E_3$

(c) $E_1 = E_2 = E_3$ (d) E_1, E_2 and E_3 cannot be compared

Q10. A particle is moving in an inverse square field If the total energy of the particle is positive, then trajectory of particle is

(a) circular (b) elliptical (c) parabolic (d) hyperbolic

Q11. In a central force field, the trajectory of a particle of mass m and angular momentum J in plane polar coordinates is given by,

$$\frac{1}{r} = \frac{m}{J^2}(1 + \varepsilon \cos\theta)$$

where, ε is the eccentricity of the particle's motion. Which one of the following choice for ε gives rise to a parabolic trajectory?

(a) $\varepsilon = 0$ (b) $\varepsilon = 1$ (c) $0 < \varepsilon < 1$ (d) $\varepsilon > 1$

Q12. Match the following List I with List II.

List I	**List II**
A. Eccentricity $e > 0$, energy $(E) > 0$	1. Circular orbit
B. Eccentricity $e = 1$ $E = 0$	2. Elliptical orbit
C. $e < 1, E < 0$	3. Parabolic orbit
D. $e = 0$, $E = -\frac{mk^2}{2J^2}$	4. Hyperbolic orbit

	A	B	C	D
(a)	1	2	3	4
(b)	2	1	3	4
(c)	3	4	1	2
(d)	4	3	2	1

Q13. A satellite in a circular orbit about the earth has a kinetic energy E_K. What is the minimum amount of energy to be added, so that it escape from the earth?

(a) $\frac{E_K}{4}$ (b) $\frac{E_K}{2}$ (c) E_K (d) $2E_K$

Q14. Two particles of identical mass move in circular orbits under a central potential $V(r)=\frac{1}{2}kr^2$. Let l_1 and l_2 be the angular momenta and r_1, r_2 be the radii of the orbits respectively. If $\frac{J_1}{J_2}$, the value of $\frac{r_1}{r_2}$ is:

(a) $\sqrt{2}$ (b) $1/\sqrt{2}$ (c) 2 (d) 1 / 2

Q15. A planet of mass *m* moves in the gravitational field of the Sun (mass *M*). If the semi-major and semi-minor axes of the orbit are *a* and *b* respectively, the angular momentum of the planet is

(a) $\sqrt{2GMm^2(a+b)}$ (b) $\sqrt{2GMm^2(a-b)}$ (c) $\sqrt{\frac{2GMm^2ab}{a-b}}$ (d) $\sqrt{\frac{2GMm^2ab}{a+b}}$

NAT (Numerical Answer Type)

Q16. If a planet revolves round the sun in a circular orbit of radius a with a period of revolution T, then (k being a positive constant) if $T\propto a^{\alpha}$ then will value of α is given by ……..

Q17. Consider a satellite going round the earth in a circular orbit at a height of $2R$ from the surface of the earth, where R is the radius of the earth. The speed of the satellite is …….$\left(\frac{gR}{\beta}\right)^{\alpha}$ then value of α..... and value of β is ……

Q18. A body of mass m moves under the action of a central force with potential $V(r)=Ar^3(A>0)$ where r about the origin. Then kinetic energy will the orbit be a circle of radius R is αAR^3 what is value of α _____?

Q19. A comet moves in an elliptical orbit with an eccentricity of $e=0.20$ around a star. The distance between the perihelion and the aphelion is $1.0\times10\,km$. If the speed of the comet at perihelion $60\ km/s$ then the speed of the comet at the aphelion is……. km/s

Q20. The acceleration due to gravity (g) on the surface of Earth is approximately 2.6 times that on the surface of Mars. Given that the radius of Mars is about one half the radius of Earth, the ratio of the escape velocity on Earth to that on Mars is approximately …….

MSQ (Multiple Select Questions)

Q21. A satellite is in elliptical orbit the earth (radius $= 6400\ km$). At perigee it has an altitude of $1100\ km$ and the apogee its apogee its altitude is $4100\ km$.

(a) Then The major axis of the orbit is $18,000\ km$

(b) Then The major axis of the orbit is $9,000\ km$

(c) The eccentricity is 0.16

(d) Eccentricity is 0.33

Q22. A planet (mass M_P) is moving around the sun (mass M_S) in an elliptical orbit of semimajor axis a. Total energy of the planet is

(a) $-\frac{GM_PM_S}{4a}$ (b) $-\frac{2GM_PM_S}{a}$ (c) $\frac{GM_PM_S}{2a}$ (d) $-\frac{GM_PM_S}{2a}$

Q23. Consider the following statements for a particle moving in an elliptic orbit under the influence of a central force.

(a) The radius vector covers equal area in equal time.

(b) The motion takes place in a plane.

(c) The angular momentum is a constant of motion.

(d) The energy remains constant and negative

Q24. Consider the following expressions n respect of earth's motion around the sun in a circular orbit of radius r with a linear speed v and angular speed ω. Then which of the following expressions are correct?

(a) v is proportional to $r^{\frac{-1}{2}}$ (b) v is proportional to $r^{\frac{-3}{2}}$

(c) $\omega\, v$ is proportional to $r^{\frac{-5}{2}}$ (d) ω is proportional to $r^{\frac{-3}{2}}$

Solutions

MCQ (Multiple Choice Questions)

Ans. 1: (b)

Solution: closest from the sun $= r_1$, speed $= v_1$

Farthest from the sun $= r_2$

According to conservation of momentum $mr_1v_1 = mr_2v_2$

$$v_2 = \frac{r_1 v_1}{r_2}$$

Ans. 2: (c)

Solution: if reduce mass $\mu = \frac{mM}{m+M}$ then $\frac{1\mu v^2}{2} = 0 - \left(-\frac{GMm}{d}\right) \Rightarrow v^2 = \frac{2G(M+m)}{d}$ (acceleration due to gravity upward from the earth)

$$v = \sqrt{\frac{2G(M+m)}{d}}$$

Ans. 3: (a)

Solution: For circular motion

$$\frac{mv^2}{r} = \frac{k}{r^2} \Rightarrow K = \frac{mv^2}{2} = \frac{k}{r^2} \qquad E = K + U \qquad U = -\frac{k}{r} \qquad E = -\frac{k}{2r}$$

Ans. 4: (d)

Solution: $V_{effctive} = \frac{J^2}{2mr^2} - \frac{k}{r} \Rightarrow \frac{dV_{effect}}{dr} = -\frac{J^2}{mr^3} + \frac{k}{r^2} = 0$ at $r = r_0$

so $J = \sqrt{r_0 km}$

Ans.5: (a)

Solution: By conservation law of energy

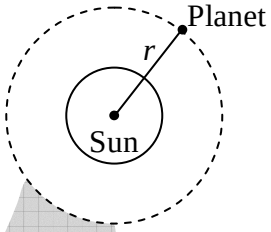
$$\Rightarrow \qquad dW = (PE)_A - (PE)_B \qquad \text{[By equation (i)]}$$

$$= \left(-\frac{GMm}{r_A}\right) - \left(-\frac{GMm}{r_B}\right) = \frac{GMm}{T_B} - \frac{GMm}{T_A} = \frac{GMm}{r_B}\left[1 - \frac{r_B}{r_A}\right]$$

Ans. 6: (c)

Solution: If a planet revolve with velocity v radius r then the centripetal force it gives as

$$F = \frac{mv^2}{r} \qquad \text{(i)}$$



The attractive M between sun and planet is given as

$$F = -\frac{k}{r^n} \qquad \text{(ii)}$$

k being a constant

By equations (i) and (ii) $\frac{k}{r^n} = \frac{mv^2}{r} \Rightarrow v^2 = \frac{k}{mr^{n-1}} \Rightarrow v = \left(\frac{k}{mr^{n-1}}\right)^{\frac{1}{2}}$

Hence, time period $= \frac{2\pi r}{v} = \frac{2\pi r}{\left(\frac{k}{mr^{n-1}}\right)^{\frac{1}{2}}} = 2\pi r\left(\frac{mr^{n-1}}{k}\right)^{\frac{1}{2}} = \frac{2\pi r.r^{\frac{n-1}{2}}\sqrt{m}}{\sqrt{k}} = 2\pi\sqrt{\frac{m}{k}}\, r^{\frac{n+1}{2}}$

Hence, time period is proportional to $r^{\frac{n+1}{2}}$

Ans. 7: (c)

Solution: Kepler's second law states that the areal velocity swept out by a radius drawn from sun to a planet is constant, i.e.,

$$\frac{dA}{dt} = \text{constant}$$

Ans. 8: (c)

Solution: Potential at earth surface $V = -\frac{GM_e}{R_e}$ (i)

Mass of earth $M_e = \text{volume} \times \text{density} = \frac{4}{3}\pi R_e^3 \rho$ (ii)

By equation (i) and (ii) becomes

$$V = -\frac{G}{R_e}\left(\frac{4}{3}\pi R_e^3 \rho\right) = -\frac{4}{3}\pi R^2 \rho G$$

Ans. 9: (b)

Solution: The eccentricity of the ellipse is given as

$$e = \sqrt{1 + \frac{2EJ^2}{Mk^2}}$$

where $k =$ constant

$M =$ mass of the object

$E =$ mechanical energy of the object eccentricity which is negative

But $e_1 < e_2 < e_3$ given in question

Then by equation (iii), $E_1 > E_2 > E_3$

Ans. 10: (c)

Solution: $\varepsilon = \left(1 + \frac{2EJ^2}{mk^2}\right)^{\frac{1}{2}} \Rightarrow \varepsilon > 1 \because \left(\frac{J}{k}\right)^2$ is positive

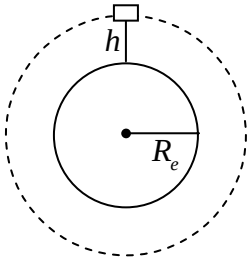
Ans. 11: (b)

Solution: $\frac{l}{r} = \frac{m}{J^2}\left(1 + \varepsilon\cos\theta\right)$ for parabolic trajectory $\varepsilon = 1$.

Ans. 12: (d)

Ans. 13: (c)

Solution: The potential energy of the satellite at a height h, i.e., at distance $\left(R_e + h\right)$ from the centre of the earth is given as



$$PE = -\frac{GM_e m}{r} \Rightarrow PE = -\frac{GM_e m}{R_e + h}$$

where m is mass of satellite and M_e is mass of earth.

$$\frac{mv^2}{r} = \frac{GM_e m}{r^2} \Rightarrow mv^2 = \frac{GM_e m}{r} \Rightarrow KE = \frac{1}{2}mv^2 = \frac{GM_e m}{2r} \Rightarrow KE = \frac{GM_e m}{2r}$$

But equations, (i) and (ii), we get

$$KE = -\frac{1}{2}PE \Rightarrow PE = -2\left(KE\right) \Rightarrow \quad PE = -2E_K \quad \text{(given } KE = E_K\text{)}$$

The total energy of satellite $= PE + KE = -2E_K + E_K = -E_K$

The total energy $(KE + PE)$ of satellite is negative and is called the binding energy of the satellite

Satellite becomes free, if its total energy is non negative

Hence, in order to escape from the earth, the minimum amount of energy to be supplied is E_K

Ans. 14: (a)

Solution: $V_{eff} = \frac{J^2}{2mr^2} + \frac{1}{2}kr^2$ where J is angular momentum.

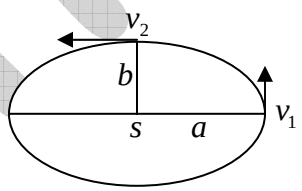
Condition for circular orbit $\frac{\partial V_{eff}}{\partial r} = 0 \Rightarrow -\frac{J^2}{mr^3} + kr = 0 \Rightarrow J^2 \propto r^4 \Rightarrow J \propto r^2$.

Thus $\frac{J_1}{J_2} = \left(\frac{r_1}{r_2}\right)^2 \Rightarrow \frac{r_1}{r_2} = \sqrt{\frac{J_1}{J_2}} \Rightarrow \frac{r_1}{r_2} = \sqrt{2}$ since $\frac{J_1}{J_2} = 2$.

Ans. 15: (d)

Solution: Assume Sun is at the centre of elliptical orbit.

Conservation of energy $\frac{1}{2}mv_1^2 - \frac{GMm}{a} = \frac{1}{2}mv_2^2 - \frac{GMm}{b}$



Conservation of momentum $L = mv_1a = mv_2b$

$$v_2 = v_1\left(\frac{a}{b}\right)$$

$$\frac{1}{2}mv_1^2 - \frac{1}{2}mv_2^2 = \frac{GMm}{a} - \frac{GMm}{b} \Rightarrow \frac{1}{2}m\left(v_1^2 - v_1^2\frac{a^2}{b^2}\right) = GMm\left(\frac{b-a}{ab}\right)$$

$$\frac{1}{2}mv_1^2\left(\frac{b^2-a^2}{b^2}\right) = GMm\left(\frac{b-a}{ab}\right) \Rightarrow \frac{1}{2}mv_1^2 = GMm\left(\frac{b}{a}\right)\cdot\frac{1}{(b+a)}$$

$$v_1 = \sqrt{2GM\left(\frac{b}{a}\right)\cdot\frac{1}{(b+a)}}$$

$$L = mv_1a = m\sqrt{2GM\left(\frac{b}{a}\right)\cdot\left(\frac{1}{b+a}\right)}\cdot a = m\sqrt{\frac{2GMab}{(b+a)}} \Rightarrow L = \sqrt{\frac{2GMm^2ab}{a+b}}$$

NAT (Numerical Answer Type)

Ans. 16: 1.5

Solution: The Kepler's third law states that square of time perk planet is directly proportional to the cube of the set axis a

i.e., $T^2 \propto a^3$

$T \propto a^{\frac{3}{2}}$

Ans. 17: $\alpha = 0.5,\ \beta = 3$

Solution: Centripetal force $= \frac{mv^2}{r}$

Newton's law of gravitation $= \frac{GMm}{r^2}$

From equation (i) and (ii), in equilibrium

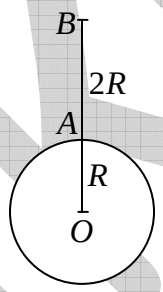
$$\frac{mr^2}{r} = \frac{GMm}{r^2}$$

$$V = \frac{\sqrt{GM}}{r}$$

$g = \frac{GM}{R^2}$ (acceleration due to gravity)

$$GM = gR^2$$

$$v = \sqrt{\frac{gR^2}{r}} = \sqrt{\frac{gR^2}{R+2R}} = \frac{\sqrt{gR^2}}{3R} = \left(\frac{gR}{3}\right)^{1/2}$$



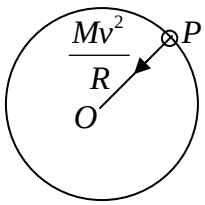
Ans. 18: 1.5

Solution: If U is potential energy then applied force F is given by

$$F = -\frac{dU}{dR} = -\frac{d}{dr}\left(Ar^3\right) \text{ at } r = R$$

$\Rightarrow$ $F = -3AR^2$ [Since]

This force is equal to centripetal force $\frac{mv^2}{R}$



i.e. $\frac{mv^2}{R} = F \Rightarrow \frac{mv^2}{R} = 3MAR^2 \Rightarrow mv^2 = 3AR^3$

$\frac{1}{2}mv^2 = \frac{3}{2}AR^3 \Rightarrow KE = \frac{3}{2}AR^3$

Ans. 19: 40

Solution: According to conservation of momentum is $mv_p r_p = mv_a r_a$

$2a = 10km \quad a = 5km \quad r_p = a(1-e) = 4km \quad r_a = a(1+e) = 6$

$v_a = \frac{r_p v_p}{r_a} = \frac{4 \times 60}{6} = 40km/s$

Ans. 20: 2.3

Solution: Escape velocity $= \sqrt{2gR}$

$\frac{\text{Escape velocity of Earth}}{\text{Escape velocity of Mass}} = \sqrt{\frac{g_e R_e}{g_m R_m}} = 2.3$ where $\frac{R_e}{R_m} = 2$ and $\frac{g_e}{g_m} = 2.6.$

MSQ (Multiple Select Questions)

Ans. 21: (a) and (c)

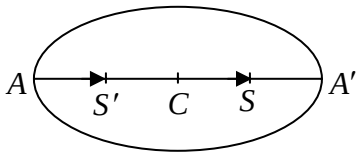
Solution: Major axis $2a$ = maximum distance + minimum distance

Maximum distance = maximum altitude + radius of earth $= (4100 + 6400)$

Minimum distance from centre of earth $= 6400 + 1100$

So, $2a = (4100 + 6400) + (6400 + 1100) = 18000\ km$

$e = \frac{r_{max-} r_{min}}{r_{max+} r_{min}} = \frac{1}{6}$



Ans. 22: (d)

Solution: $E = -\frac{k}{2a} = -\frac{GM_P M_s}{2a}$

Ans. 23: (a), (b), (c) and (d)

Solution: Angular momentum $J = r \times mv$ implies that angular momentum is perpendicular to r and v.

Hence, the path of particle under the influence of a central force must lie in a plane.

Torque $\tau = \dfrac{dJ}{dt} = r \times \vec{F} = r \times rf\left(r\right) = 0$

Thus, angular momentum $J = \text{constant}$

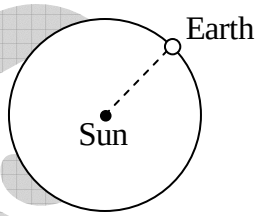
Areal velocity $\dfrac{1}{2}\vec{r} \times \vec{v} = \dfrac{1}{2} \times \dfrac{J}{m} = \dfrac{J}{2m} = \text{constant}$ $\left(J = mvr\right)$

The radius vector of orbit sweeps out equal area in equal interval of time.

Ans. 24: (a) and (d)

Solution: It is supposed that earth revolves around the sun in orbit of radius r with linear velocity v

Centripetal force $= \dfrac{mv^2}{r}$ (i)



Where $m =$ mass of the earth.

The attractive force towards sun is given as

$$F = \frac{GM_3 m}{r^2} \quad \text{(ii)}$$

$M_1 =$ mass of the sun

by equation (i) and (ii)

$$\frac{GM_1 m}{r^2} = \frac{mv^2}{r}$$

$$\Rightarrow \quad v^2 = \frac{GM_1}{r}$$

$$\Rightarrow \quad v = \frac{\sqrt{GM_1}}{\sqrt{r}} \quad \text{(iii)}$$

Since $G =$ gravitational constant hence, $\Rightarrow v \propto r^{-\frac{1}{2}}$ (iv)

First statement is true.

If ω is angular velocity, then

$$v = \omega r \Rightarrow \quad \omega = \frac{v}{r} \Rightarrow \quad \omega = \left(\frac{\sqrt{GM}}{r}\right)\frac{1}{r} \Rightarrow \omega \propto r^{-\frac{3}{2}} \quad \text{(v)}$$

Hence equation (i) $v \propto r^{-\frac{1}{2}}$ (ii) $\omega \propto r^{-\frac{3}{2}}$

4. Moment of Inertia Rotational Dynamics

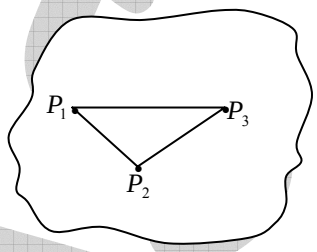
4.1 Rigid Body Dynamics

A rigid body is defined as system of particles in which the distance between any two particles remains fixed throughout the motion. Degree of freedom of rigid body.

To define rigid body there must be minimum 3 non-collinear point.

Let $p_1(x_1, y_1, z_1)$, $p_2(x_2, y_2, z_2)$, $p_3(x_3, y_3, z_3)$ is there three point.

So equation of constrained is



$$r_{12} = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 = c_1$$

$$r_{23} = (x_2 - x_3)^2 + (y_2 - y_3)^2 + (z_2 - z_3)^2 = c_2$$

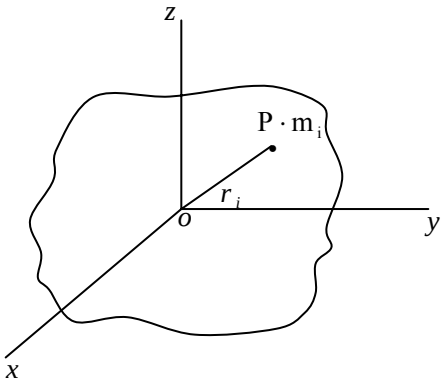
$$r_{13} = (x_1 - x_3)^2 + (y_1 - y_3)^2 + (z_1 - z_3)^2 = c_3$$

$$Dof = 3N - k$$

$3 \times 3 - 3 = 6$ So there is six degree of freedom for rigid body

4.1.1 Angular Momentum and Inertia Tensor.

Let us consider the motion of a rigid body rotating about a fixed point o in the body as shown in figure.



At any instant of time, the body will rotating with velocity ω about the instantaneous through o.

A particle P of the body, having the position vector r_1 with respect to o has instantaneous velocity v_i relative to o, given by

$\vec{v}_i = \vec{\omega} \times \vec{r}_i$, where $\vec{\omega} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

Angular momentum of Point P about the point o is given by

$$\vec{J}_p = \vec{r}_i \times m_i \vec{v}_i$$

$$J_P = \vec{r}_i \times m_i (\vec{\omega} \times \vec{r}_i)$$

$$J = \sum_{i=1}^{N} \vec{r}_i \times m_i \vec{v}_i = \sum_{i}^{N} m_i \vec{r}_i \times (\vec{\omega} \times \vec{r}_i)$$

$$J = \sum_i m_i \left[r_i^2 \omega - (r_i \omega) r_i \right]$$

$$\begin{pmatrix} J_x \\ J_y \\ J_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{zy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

$I_\alpha = \sum_\beta I_{\alpha\beta} \omega_\beta$ $\quad$ $I_{\alpha\beta}$ is inertia tensor

where $I_{xx} = \sum_i m_i (y_i^2 + z_i^2)$, $I_{yy} = \sum_i m_i (x_i^2 + z_i^2)$, $I_{zz} = \sum_i m_i (x_i^2 + y_i^2)$ known as moment of inertia and

$$I_{xy} = I_{yx} = -\sum_i m_i x_i y_i, \; I_{xz} = I_{zx} = -\sum_i m_i x_i z_i, \; I_{yz} = I_{zy} = -\sum_i m_i y_i z_i$$

is known as product of inertia.

$$I_{\alpha\beta} = I_{\beta\alpha} = \sum_{i=1}^{N} m_i \left[\delta_{\alpha\beta} r_i^2 - x_{i\alpha} x_{i\beta} \right]$$

where $\alpha = 1,2,3$ $\beta = 1,2,3$ and one can denote x, y, z by x_1, x_2, x_3 respectively.

For continuous system $I_{\alpha\beta}$ is reduce to

$\int dm \left[\delta_{\alpha\beta} x_i^2 - x_{i\alpha} x_{i\beta} \right]$ where dm is elemental mass.

4.1.2 Principal Moment of Inertia:

If one can diagonalized Inertia tensor into diagonal matrix then the diagonal element is known as principal moments of inertia

and x, y, z component of angular momentum reduce to

$$J_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$$

$$J_y = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z$$

$$J_z = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z$$

$$\begin{pmatrix} I_{xx}-I & I_{xy} & I_{xz} \\ I_{xy} & I_{yy}-I & I_{yz} \\ I_{xz} & I_{yz} & I_{zz}-I \end{pmatrix}\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = 0$$

characteristic equation $I = I_1, I_2, I_3$ are three principal moment of inertia.

Rotational kinetic energy of a rigid body

$$\vec{v}_i = \vec{\omega} \times \vec{r}_i$$

Kinetic energy of particle m_i (which is explained previous topic).

$$T_i = \frac{1}{2} m_i v_i^2 = \frac{1}{2} m_i \vec{v}_i \cdot \vec{v}_i$$

Kinetic energy for entire body

$$T = \sum \frac{1}{2} m_i v_i^2 = \sum_i \frac{1}{2} m_i \vec{v}_i \cdot \vec{v}_i \; \frac{1}{2}\sum \left(\vec{\omega} \times \vec{r}_i\right) \cdot m_i \vec{v}_i = \frac{1}{2}\sum_i \vec{\omega} \cdot \left(\vec{r}_i \times m_i \vec{v}_i\right) = \frac{1}{2}\vec{\omega} \cdot \sum_i \left(\vec{r}_i \times m\vec{v}_i\right)$$

$$J = \sum_i \left(\vec{r}_i \times m_i \vec{v}_i\right) \text{ so } T = \frac{1}{2}\vec{\omega} \cdot \vec{J}$$

Kinetic energy in a co-ordinate system of principle axis is given by

$$T = \frac{1}{2} I_x \omega_x^2 + \frac{1}{2} I_y \omega_y^2 + \frac{1}{2} I_z \omega_z^2$$

Example: Consider a cube of volume a^3 and mass M which is situated such the origin o is at one of the corner consider cube has uniform density ρ.

(a) Find the moment of inertia tensor.

(b) Find the principal moment of inertia tensor.

Solution: (a) $I_{xx} = \int \rho\left(y^2+z^2\right)dV = \int_0^a\int_0^a\int_0^a \rho\left(y^2+z^2\right)dx\,dy\,dz = \frac{2\rho a^5}{3}$

$$\Rightarrow I_{xx} = \frac{2Ma^5}{a^3 3} = \frac{2M}{3}a^2 \text{ where } \rho = \frac{M}{a^3}$$

Similarly,

$$I_{yy} = \int_0^a\int_0^a\int_0^a \rho\left(x^2+z^2\right)dV = \frac{2}{3}Ma^2 \quad \text{and} \quad I_{zz} = \int_0^a\int_0^a\int_0^a \rho\left(x^2+y^2\right)dV = \frac{2}{3}Ma^2$$

$$I_{xy} = -\int_0^a\int_0^a\int_0^a \rho xy\,dx\,dy\,dz = I_{yx} = -\frac{Ma^2}{4} \quad \text{and} \quad I_{xz} = I_{zx} = -\int_0^a\int_0^a\int_0^a \rho xz\,dV = -\frac{1}{4}Ma^2$$

$$I_{yz} = I_{yz} = -\int_0^a\int_0^a\int_0^a \rho yz\,dV = -\frac{1}{4}Ma^2$$

Hence $I = \frac{Ma^2}{12}\begin{pmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{pmatrix}$.

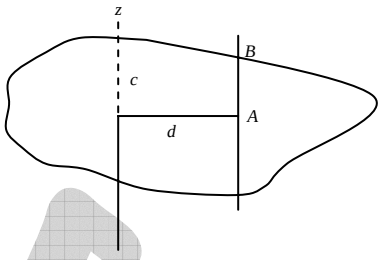
(b) Principle M.I. is given by characteristic equates

$$\begin{pmatrix} \frac{8Ma^2}{12}-I & -\frac{3Ma^2}{12} & -\frac{3Ma^2}{12} \\ -\frac{3Ma^2}{12} & \frac{8Ma^2}{12}-I & -\frac{3Ma^2}{12} \\ -\frac{3Ma^2}{12} & -\frac{3Ma^2}{12} & \frac{8Ma^2}{12}-I \end{pmatrix} = 0$$

$$\Rightarrow I_1 = I_2 = \frac{11}{12}Ma^2 \quad \text{and} \quad I_3 = \frac{1}{6}Ma^2$$

4.1.3 Theorem of Parallel Axes:

Suppose we have to obtain the moment of inertia of a body about a given line AB. Let C be the centre of mass of the body and CZ be the line parallel to AB through C. Let I and I_c be the moment of inertia of the body about AB and CZ respectively. Then



$I = I_c + md^2$ where d is the perpendicular distance between lines AB and CZ and m is mass of the body.

4.1.4 Theorem of Perpendicular Axes:

If I_x, I_y and I_z is principle moment of inertia about x, y, z axis respectively, then

$$I_z = I_x + I_y$$

4.2 Particular Cases of Moment of Inertia

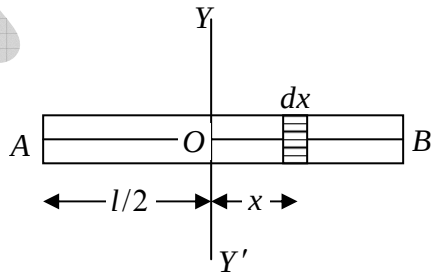
Let us now see how to obtain expressions for moments of inertia in some typical and important cases, with regular geometrical shapes.

4.2.1. Moment of Inertia of a Uniform Rod

(i) about an axis through its centre and perpendicular to its length.

Let AB be a thin uniform rod, of length l and mass M, free to rotate about an axis YOY' passing through its centre O and perpendicular to its length, as shown in figure. Since the rod is uniform, its mass per unit length is clearly M/l.

Considering a small element of the rod, of length dx at a distance x from the axis through O, we have mass of the element $= (M/l)\cdot dx$ and therefore, its moment of inertia about the axis (YOY') through $O = (M/l)\cdot dx \cdot x^2$.



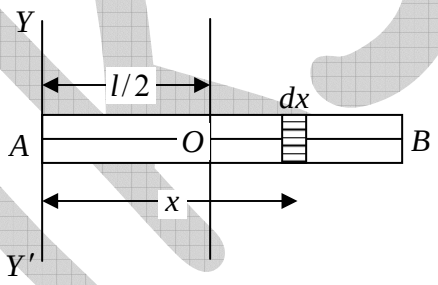
The moment of inertia (l) of the whole rod about the axis YOY' is thus given by the integral of the above expression between the limits $x = -l/2$ and $x = +l/2$ or by twice its integral between the limits $x = 0$ and $x = l/2$, i.e.

$$I = 2\int_0^{l/2} \frac{M}{l} x^2 dx = \frac{2M}{l}\left[\frac{x^3}{3}\right]_0^{l/2} = \frac{2M}{l}\cdot\frac{l^3}{24} = \frac{Ml^2}{12}$$

(ii) About an axis through one end of the rod and perpendicular to its length.

Proceeding as in case (i) above, we obtain the moment of inertia of the rod about the axis, now passing through one end A of the rod, by integrating the expression for the moment of inertia of the element dx of the rod, between the limits $x = 0$ at A and $x = l$ at B, i.e. ,

$$I = \int_0^{l} \frac{M}{l} x^2 dx = \frac{2M}{l}\cdot\frac{l^3}{24} = \frac{Ml^2}{12}$$



Alternatively, we could obtain the same result by an application of the principle of parallel axes, according to which

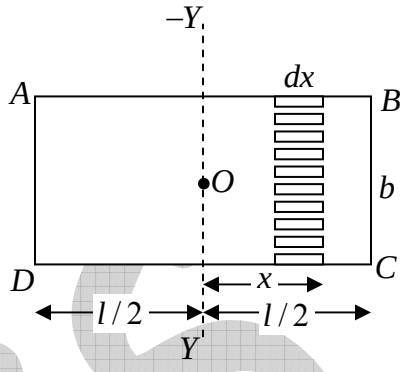
M.I. of she rod about the axis YAY' = its M.I. about a parallel axis through O + (mass of the rod × square of the distance between the two axes).

So that, $I = \frac{Ml^2}{12} + M\left(\frac{l}{2}\right)^2 = \frac{Ml^2}{12} + \frac{Ml^2}{4} = \frac{Ml^2}{3}$

4.2.2 Moment of Inertia of a Rectangular Lamina (or bar)

(i) About an axis through its centre and parallel to one side. Let $ABCD$ be a rectangular lamina, of length l, breadth b and mass M and let YOY' be the axis through its centre O and parallel to the side AD or BC about which its moment of inertia is to be determined.

Consider an element, or a small rectangular strip of the lamina, parallel to, and at a distance x from the axis. The area of this strip or element $= dx \times b$. And, since the mass per unit area of the lamina $= \dfrac{M}{(l \times b)}$, we have



mass of the strip or element $= \dfrac{M}{l \times b} \times dx \times b = \dfrac{M}{l} dx$.

And therefore, M.I. of the element about the axis YOY'

$$= \frac{M}{l} dx \cdot x^2$$

The moment of inertia (I) of the whole rectangular lamina is then given by twice the integral of the above expression between the limits $x = 0$ and $x = \dfrac{l}{2}$.

i.e., $$I = 2\int_0^{l/2} \frac{M}{l} x^2 dx = \frac{2M}{l} \int_0^{l/2} x^2 dx = \frac{2M}{l} \left[\frac{x^2}{3} \right]_0^{l/2}$$

$$= \frac{2M}{l} \cdot \frac{l^3}{24} = \frac{Ml^2}{12}$$

As will be readily seen, if b be small, the rectangular lamina becomes a rod of length l whose M.I. about the axis YOY' passing through its centre and perpendicular to its length would be $\dfrac{Ml^2}{12}$, as obtained under case I, (i) above.

(ii) About one side. in this case, since the axis coincides with AD or BC, we integrate the expression for the M. I. of the element of length dx and distance x from the axis, i.e. $\left(\dfrac{M}{l}\right) dx \cdot x^2$ between the limits $x = 0$ at AD and $x = l$ at BC. So that M. I. of the lamina about side AD or BC is given by

$$I = \int_0^l \frac{M}{l} x^2 dx = \frac{M}{l} \left[\frac{x^2}{3} \right]_0^l = \frac{Ml^2}{3}$$

Alternatively, by the principle of parallel axes, $I = \frac{Ml^2}{12} + M\left(\frac{l}{2}\right)^2 = \frac{Ml^2}{3}$.

This is again the same case as that of the M.I. of a rod about an axis passing through one of its ends and perpendicular to its length [case I (ii)], for, as pointed out above, if b be small, the rectangular lamina too reduces to a thin rod of length l.

(iii) About an axis passing through its centre and perpendicular to its plane. This may be easily obtained by an application of the principle of perpendicular axes to case (i) above, according to which, M. I. of the lamina about an axis through O and perpendicular to its plane = M.I. of the lamina about an axis through O parallel to b + M.I. of the lamina about an axis through O parallel to l i.e.,

$$I = \frac{Ml^2}{12} + \frac{Mb^2}{12} = \frac{M\left(l^2 + b^2\right)}{12}$$

This relation is true for both thin (i.e., laminar) as well as thick plates or bars, because no stipulation has been made in deducing it as to the thickness of the lamina. And, in fact, a thick rectangular plate or bar may be regarded as a combination of thin or laminar plates or bars, piled up one over the other.

(iv) About an axis passing through the mid-point of one side and perpendicular to its plane. In this case the axis passes through the mid-point of side AB or BC, say, and perpendicular to the plane of the lamina, so that it is parallel to the axis through O (the c.m. of the lamina) in case (iii).

In accordance with the principle of parallel axes, therefore, the M. I. of the lamina about this axis is given by

$$I = M\frac{\left(l^2 + b^2\right)}{12} + M\left(\frac{l}{2}\right)^2 = M\left(\frac{l^2 + b^2}{12} + \frac{l^2}{4}\right) = M\left(\frac{l^2}{3} + \frac{b^2}{12}\right)$$

And, if the axis passes through the mid-point of AB or DC, we, similarly have

$$I = M\left(\frac{l^2 + b^2}{12}\right) + M\left(\frac{b}{2}\right)^2 = M\left(\frac{l^2 + b^2}{12} + \frac{b^2}{4}\right) = M\left(\frac{l^2}{12} + \frac{b^2}{3}\right)$$

(v) About an axis passing through one of its corners and perpendicular to its plane. Let the axis pass through the corner D of the lamina. Since it is perpendicular to the plane of the lamina, it is parallel to the axis through its centre of mass O in case (iii). Again, therefore, by the principle of parallel axes, we have moment of inertia of the rectangular lamina about this axis through D given by

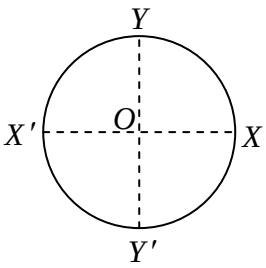
$$I = M\frac{\left(l^2+b^2\right)}{12} + Mr^2,$$

where r is the distance between the two axes. Clearly, $r^2 = (l/2)^2 + (b/2)^2 = \left(l^2+b^2\right)/4$.

So that, $I = M\left(\frac{l^2+b^2}{12}\right) + M\left(\frac{l^2+b^2}{4}\right) = M\left(\frac{l^2+b^2+3l^2+3b^2}{12}\right) = M\left(\frac{l^2+b^2}{3}\right)$

4.2.3 Moment of inertia of a hoop or a thin circular ring.

(i) About an axis through its centre and perpendicular to its plane. Let the radius of the hoop or the thin circular ring be R and its mass M.



Consider a particle of mass m of the hoop or the ring. Clearly, its M.I. about an axis through the centre O of the hoop or the ring and perpendicular to its plane $= mR^2$.

$\therefore$ M.I. of the entire hoop or ring about this axis passing through its centre and perpendicular to a plane, $I = \Sigma mR^2 = MR^2$ [$\Sigma m = M$, the mass of the hoop or ring]

(ii) About its diameter. Obviously, due to symmetry, the M.I. of the hoop or the ring will be the same about one diameter as about another. Thus, if I be its M.I. about the diameter XOX' (Figure), it will also be I about the diameter YOY' perpendicular to XOX'.

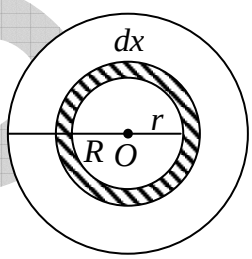
By the principle of perpendicular axe. therefore, the M.I. of the hoop or the ring about the axis through its centre O and perpendicular to its plane is equal to the sum of its moments of inertia also the perpendicular axes XOX' and YOY' in its own plane and intersecting at O, i.e.,

$$I + I = MR^2 \text{ or } 2I = MR^2 \text{ whence, } I = \frac{MR^2}{2}.$$

4.2.4 Moment of inertia of a circular lamina or disc

(i) About an axis through its centre and perpendicular to its plane:

Sine as we know, $I = MK^2 = MR^2$, we have here $MK^2 = MR^2$ whence, $K = R$, i e., the radius of the hoop or the ring is equal to its radius of gyration about the axis through its centre and perpendicular to its plane. This gives us another definition of the radius of gyration of a body about a given axis, viz., that it is numerically equal to the radius of a hoop or a circular ring, infinitely thin, of the same mass as the body and having the same moment of inertia about the axis passing through its centre and perpendicular to its plane as the body has about the given axis.



Let M be the radius of the disc and R, its radius, so that its mass per unit area is equal to $\frac{M}{\pi R^2}$.

Considering a ring of the disc, of width dx, and distance x from the axis passing through O and perpendicular to the plane of the disc, we have

Area of the ring =circumference $\times$ width $= 2\pi x dx$,

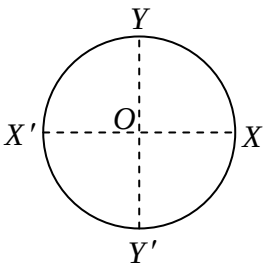
and hence its mass

$$= \left(\frac{M}{\pi R^2}\right) \times 2\pi x dx = \frac{2Mxdx}{R^2}$$

Since the whole disc may be supposed to be made up of such like concentric rings of radius ranging from 0 to R, the M.I. of the whole disc about the axis through O and perpendicular to its plane, i.e. I, obtained by integrating the above expression for the M.I. of the ring, between the limits $x = 0$ and $x = R$. Thus,

$$I = \int_0^R \frac{2Mx}{R^2} x^2 dx = \frac{2M}{R^2}\left[\frac{x^4}{4}\right]_0^R = \frac{2M}{R^2} \times \frac{R^4}{4} = \frac{MR^2}{2}.$$

(ii) About a Diameter. Here, again, due to symmetry, the M.I. of the disc, about one diameter is the same as about another. So that, if I be the M.I. of the disc about each of the perpendicular diameters XOX' and YOY', (Figure), we have, by the principle of perpendicular axes,

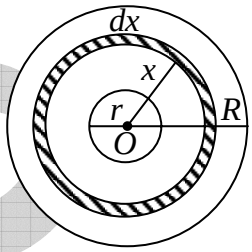


$I + I = M.I.$ of the disc about an axis through O and perpendicular to its plane,

i.e. $2I = \frac{MR^2}{2}$, whence, $I = \frac{MR^2}{4}$.

4.2.5 Moment of Inertia of an Annular Ring or Disc:

(i) About an axis through its centre and perpendicular to its plane. An annular disc is just an ordinary disc, with a smaller coaxial disc removed from it, leaving a concentric circular hole in it, as shown in figure.



If R and r be the outer and inner radii of the disc and its mass M, we have mass per unit area of the disc $\frac{M}{\pi}\left(R^2 - r^2\right)$.

Now, the disc may be imagined to be made up of a number of circular rings, with their radii ranging from r to R. So that, considering one such ring of radius x and width dx, we have

face area of the ring $= 2\pi x dx$ and

$$\therefore \text{ its mass } = 2\pi x dx\left[\frac{M}{\pi\left(R^2 - r^2\right)}\right] = \left[\frac{2Mx}{\left(R^2 - r^2\right)}\right]dx$$

and hence its M.I. about the axis through O and perpendicular to its plane

$$\frac{2Mx}{\left(R^2 - r^2\right)}dx \cdot x^2 = \frac{2Mx^2}{\left(R^2 - r^2\right)}dx$$

The M.I. of the whole annular disc i.e. I, is therefore, given by the integral of the above expression between the limits $x = r$ and $x = R$

$$I = \int_r^R \frac{2Mx^3}{\left(R^2 - r^2\right)}dx = \frac{2M}{\left(R^2 - r^2\right)}\int_r^R x^3 dx = \frac{2M}{\left(R^2 - r^2\right)}\left[\frac{x^4}{4}\right]_r^R$$

$$= \frac{2M}{\left(R^2 - r^2\right)}\left[\frac{\left(R^4 - r^4\right)}{4}\right] = \frac{M\left(R^2 + r^2\right)}{2}$$

(ii) About a Diameter. Due to symmetry, the M.I. of the annular disc about one diameter is the same as about another, say I. Then, clearly in accordance with the principle of perpendicular axes, the sum of its moment of inertia about two perpendicular diameters must be equal to its M.I. about the axis passing through its centre (where the two diameters intersect) and perpendicular to its plane,

i.e., $I + I = \dfrac{M\left(R^2 + r^2\right)}{2}$ or $2I = \dfrac{M\left(R^2 + r^2\right)}{2}$, whence, $I = \dfrac{M\left(R^2 + r^2\right)}{4}$.

4.2.6 Moment of Inertia of a Solid Cylinder:

(i) About its own axis of cylindrical symmetry. A solid cylinder is just a thick circular disc or a number of thin circular disc (all of the same radius) piled up one over the other, so that its axis of cylindrical symmetry is the same as the axis passing through the centre of the thick disc (or the pile of thin discs) and perpendicular to its plane

$\therefore$ M.I. of the thick disc (or the pile of thin discs) of the same mass and radius about the axis through its centre and perpendicular to its plane.

or $I = \dfrac{MR^2}{2}$

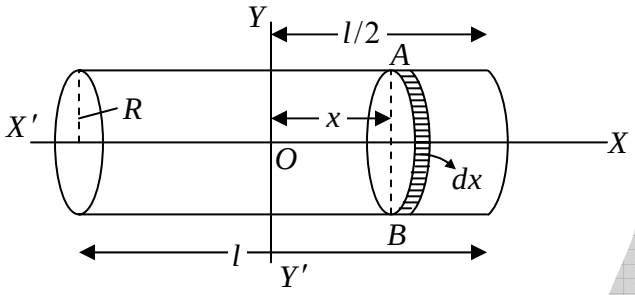
(ii) About the axis through its centre and perpendicular to its axis of cylindrical symmetry. If R be the radius, l, the length and M, the mass of the solid cylinder, supposed to be uniform and of a homogeneous composition, we have its mass per unit length = $\dfrac{M}{l}$.

Now, imagining the cylinder to be made up of a number of discs each of radius R, placed adjacent to each other, and considering one such disc of thickness dx and at a distance x from the centre O of the cylinder, (figure), we have

$$\text{Mass of the disc} = \left(\frac{M}{l}\right)dx \text{ and radius} = R$$

And $\therefore$ M.I. of the disc about its diameter $AB = \dfrac{M}{l}dx \cdot \dfrac{R^2}{4}$ and its M.I. about the parallel axis YOY', passing through the centre O of the cylinder and perpendicular to its axis of cylindrical symmetry (or its length), in accordance with the principle of parallel axes,

$$= \frac{M}{l}dx\frac{R^2}{4} + \frac{M}{l}dx \cdot x^2$$



Hence, M.I. of the whole cylinder about this axis, i.e. $I =$ twice the integral of the above expression between the limits $x = 0$ and $x = \frac{l}{2}$,

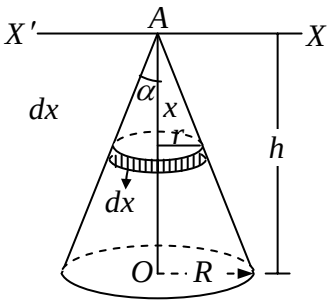
i.e., $$I = 2\int_0^{l/2}\left(\frac{M}{l}\cdot\frac{R^2}{4}dx + \frac{M}{l}x^2dx\right) = \frac{2M}{l}\int_0^{l/2}\left(\frac{R^2}{4}dx + x^2dx\right)$$

$$= \frac{2M}{l}\left[\frac{R^2x}{4} + \frac{x^3}{3}\right]_0^{l/2}$$

or $$I = \frac{2M}{l}\left[\frac{R^2}{4}\cdot\frac{l}{2} + \frac{l^3}{8\times 3}\right] = \frac{2M}{l}\left(\frac{R^2l}{8} + \frac{l^3}{24}\right) = M\left(\frac{R^2}{4} + \frac{l^2}{12}\right)$$

4.2.7 Moment of Inertia of a Solid Cone:

(i) About its vertical axis. Let M be the mass of the solid cone, h its vertical height and R, the radius of its base (figure below).



Clearly, volume of the cone $= \frac{1}{3}\pi R^2h$ and if ρ be the density of its material its mass $M = \frac{1}{3}\pi R^2h\rho$, whence, $\rho = \frac{3M}{\pi R^2h}$.

Now, the cone may be imagined to consist of a number of discs of progressively decreasing radii, from R to 0 piled up one over the other.

Considering one such disc of thickness dx and at a distance x from the vertex A of the cone, we have

Radius of the disc, $r = x\tan\alpha$,

where α is the semi-vertical angle of the cone, and, therefore its volume $= \pi r^2 dx = \pi x^2 \tan^2 \alpha dx$ and its mass $= \pi x^2 \tan^2 \alpha \rho dx$.

Hence, M.I. of the disc about the vertical axis AO of the cone (i.e. an axis passing through its centre and perpendicular to its plane) = mass $\times \dfrac{(radius)^2}{2}$

$$= \pi x^2 \rho \tan^2\alpha \, dx \cdot \frac{r^2}{2} = \frac{\pi x^2 \rho \tan^2\alpha \, dx \cdot x^2 \tan^2 \alpha}{2} = \left(\frac{\pi\rho \tan^4 \alpha}{2}\right) x^4 dx$$

And $\because$ M.I. of the entire cone about its vertical axis AO is given by

$$I = \int_0^h \frac{\pi\rho \tan^4 \alpha}{2} x^4 dx = \frac{\pi\rho \tan^4 \alpha}{2} \int_0^h x^4 dx = \frac{\pi\rho \tan^4 \alpha}{2} \left[\frac{x^5}{5}\right]_0^h$$

$$= \frac{\pi\rho \tan^4 \alpha}{2} \cdot \frac{h^5}{5} = \frac{\pi\rho R^4}{2h^4} \cdot \frac{h^5}{5},$$

substituting R/h for $\tan\alpha$

or, substituting the value of ρ obtained above, we have

$$I = \frac{\pi \cdot 3M}{\pi R^2 h} \cdot \frac{R^4}{2h^4} \cdot \frac{h^5}{5} = \frac{3MR^2}{10}$$

(ii) About an axis passing through the vertex and parallel to its base.

Again, considering the disc at a distance x from the vertex of the cone, we have its M.I. about its diameter $= \dfrac{mass \times (radius)^2}{4}$

$$= \pi x^2 \rho \tan^2 \alpha dx \cdot \frac{r^2}{4} = \frac{\pi x^2 \tan^2 \alpha \rho dx \cdot x^2 \tan^2 \alpha}{4} = \left(\frac{\pi\rho \tan^4 \alpha}{4}\right) x^4 dx$$

$\therefore$ Its M.I. about the parallel axis XX', parallel to its base is given by

$$= \left(\frac{\pi\rho \tan^4 \alpha}{4}\right) x^4 dx + \pi x^2 \tan^2 \alpha \rho x^2 dx = \left(\frac{\pi\rho \tan^4 \alpha}{4}\right) x^4 dx + \pi\rho \tan^2 \alpha x^4 dx$$

Hence M.I. of the entire cone about the axis XX', parallel to the base is given by

$$I = \int_0^h \frac{\pi\rho \tan^4}{4} x^4 dx + \pi\rho \tan^2 \alpha x^4 dx = \frac{\pi\rho \tan^4 \alpha}{4} \int_0^h x^4 dx + \pi\rho \tan^2 \alpha \int_0^h x^4 dx$$

$$= \frac{\pi\rho}{4}\cdot\frac{R^4}{h^4}\left[\frac{x^5}{5}\right]_0^h + \pi\rho\cdot\frac{R^2}{h^2}\left[\frac{x^5}{5}\right]_0^h = \frac{\pi\rho R^4}{4h^4}\cdot\frac{h^5}{5} + \frac{\pi\rho R^2}{h^2}\cdot\frac{h^5}{5} \qquad \left[\because \tan\alpha = \frac{R}{h}\right]$$

or, substituting the value of ρ, we have M.I. of the cone about the axis XX', i.e.

$$I = \frac{\pi\cdot 3M}{4\pi R^2 h}\cdot\frac{R^4}{h^4}\cdot\frac{h^5}{5} + \frac{\pi\cdot 3M}{4\pi R^2 h}\cdot\frac{R^2}{h^2}\cdot\frac{h^5}{5} = \frac{3MR^2}{20} + \frac{3Mh^2}{5}$$

4.2.8 Moment of Inertia of a Hollow Cylinder

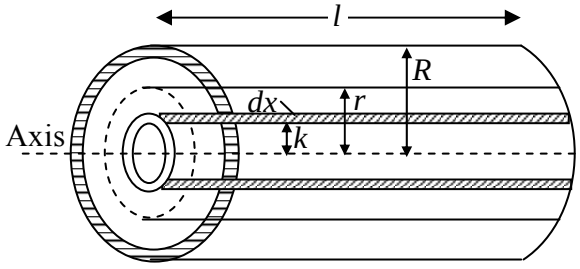
(i) About its axis of cylindrical symmetry. A hollow cylinder may be considered to be a thick annular disc or a combination of thin annular discs, each of the same external and internal radii, placed adjacent to each other, the axis of the cylinder (i.e. its axis of cylindrical symmetry) being the same as the axis passing through the centre of the thick annular disc (or the combination of thin annular discs) and perpendicular to its plane.

The M.I. of the hollow cylinder about its own axis is, therefore, the same as that of a thick annular disc (or a combination of thin annular discs) of the same mass M and external and internal radii R and r respectively about the axis passing through its centre and perpendicular to its plane.

i.e., $$I = \frac{M\left(R^2 + r^2\right)}{2}$$

Alternatively, we may obtain the same result directly as follows:

Let R and r be the external and internal radii respectively of a hollow cylinder of length l and mass M, (figure below). Then face-area of the cylinder $= \pi\left(R^2 - r^2\right)$ and its volume $= \pi\left(R^2 - r^2\right)l$ and, therefore, its mass per unit volume $= \dfrac{M}{\pi\left(R^2 - r^2\right)l}$.



Imagining the cylinder to be made up of a large number of thin, coaxial cylinders, with their radii varying from r to R and considering one such cylinder of radius x and thickness dx, we have its face area $= 2\pi x dx$ and its volume $= 2\pi x dx l$ and hence its mass $= \dfrac{2\pi x dx \cdot l \times M}{\pi\left(R^2 - r^2\right)l} = \dfrac{2Mxdx}{\left(R^2 - r^2\right)}$.

And therefore, its M.I. about the axis of the cylinder

$$= \frac{2Mxdx}{\left(R^2 - r^2\right)}x^2 = \frac{2M}{\left(R^2 - r^2\right)}x^3 dx$$

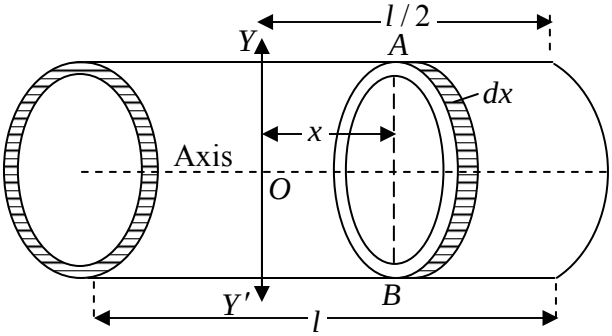
Hence, M.I. of the entire cylinder about its axis, i.e. I, is given by

$$I = \int_r^R \frac{2M}{\left(R^2 - r^2\right)}x^3 dx = \frac{2M}{\left(R^2 - r^2\right)}\int_r^R x^3 dx = \frac{2M}{\left(R^2 - r^2\right)}\left[\frac{X^4}{4}\right]_r^R$$

$$= \frac{2M}{\left(R^2 - r^2\right)}\left(\frac{R^4 - r^4}{4}\right) = M\left(\frac{R^2 + r^2}{2}\right)$$

(ii) About an axis passing through its centre and perpendicular to its own axis. Again, if R and r be the external and internal radii respectively of the hollow cylinder, I, its length and M, its mass, we have mass per unit volume of the cylinder $= \dfrac{M}{\pi\left(R^2 - r^2\right)l}$.

Imagining the hollow cylinder to be made up of a large number of annular discs, of external and internal radii R and r respectively, placed adjacent to each other, and considering one such disc at a distance x from the axis YOY' passing through the centre O of the cylinder and perpendicular to its own axis, we have surface area of the disc $= \pi\left(R^2 - r^2\right)$, volume $= \left(R^2 - r^2\right)dx$ and therefore, its mass $= \pi\left(R^2 - r^2\right)dx \times \dfrac{M}{\pi\left(R^2 - r^2\right)l} = \dfrac{Mdx}{l}$



So that, M.I. of the disc about its diameter $(AB) = \frac{Mdx}{l} \cdot \frac{\left(R^2 + r^2\right)}{4}$

Hence, M.I. of entire hollow cylinder about the axis YOY' is equal to twice the integral of the above expression between the limits $x = 0$ and $x' = \frac{l}{2}$, i.e.

$$I = 2\int_0^{l/2}\left[\frac{M\left(R^2 + r^2\right)}{4l}dx + \frac{M}{l}x^2 dx\right] = \frac{2M}{l}\int_0^{l/2}\left[\frac{R^2 + r^2}{4}dx + x^2 dx\right]$$

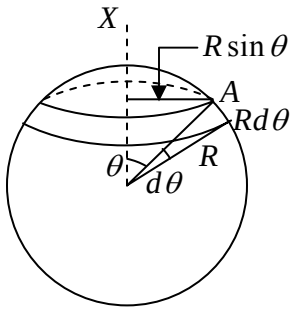
$$= \frac{2M}{l}\left[\frac{\left(R^2 + r^2\right)x}{4} + \frac{x^3}{3}\right]_0^{l/2}$$

or $$I = \frac{2M}{l}\left[\frac{\left(R^2 + r^2\right)l}{4\times 2} + \frac{l^3}{8\times 3}\right] = M\left[\frac{\left(R^2 + r^2\right)}{4} + \frac{l^2}{12}\right]$$

4.2.9Moment of inertia of a uniform hollow sphere about a Diameter

Let M and R be the mass and the radius of the sphere O its centre and OX the given axis. The mass is spread over the surface of the sphere and the inside is hollow.

Let us consider a radius OA of the sphere at an angle θ with the axis OX and rotate this radius about OX. The point A traces a circle on the sphere. Now change θ to $\theta + d\theta$ and get another circle of somewhat larger radius on the sphere. The part of the sphere between these two circles, shown in the figure, forms a ring of radius $R\sin\theta$. The width of this ring is R and its periphery is $2\pi R\sin\theta$. Hence,



the area of the ring $= (2\pi R\sin\theta)(Rd\theta)$.

Mass per unit area of the sphere $= \frac{M}{4\pi R^2}$

The mass of the ring $= \frac{M}{4\pi R^2}(2\pi R\sin\theta)(Rd\theta) = \frac{M}{2}\sin\theta\, d\theta$

The moment of inertia of this elemental ring about OX is

$$dI = \left(\frac{M}{2}\sin\theta\, d\theta\right)\left(R\sin\theta\right)^2 = \frac{M}{2}R^2\sin^3\theta\, d\theta$$

As θ increases from 0 to π, the elemental rings cover the whole spherical surface. The moment of inertia of the hollow sphere is, therefore,

$$I = \int_0^{\pi}\frac{M}{2}R^2\sin^3\theta\, d\theta = \frac{MR^2}{2}\left[\int_0^{\pi}\left(1-\cos^2\theta\right)\sin\theta\, d\theta\right]$$

$$= \frac{MR^2}{2}\left[\int_{\theta=0}^{\pi} -\left(1-\cos^2\theta\right)d\left(\cos\theta\right)\right] = \frac{-MR^2}{2}\left[\cos\theta - \frac{\cos^3\theta}{3}\right]_0^{\pi} = \frac{2}{3}MR^2.$$

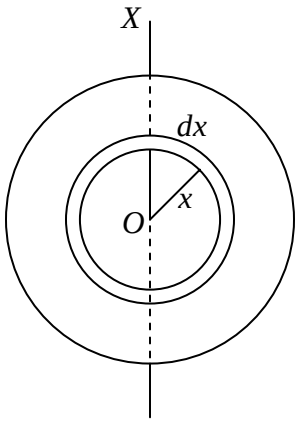
4.2.10 Moment of Inertia of a Uniform Solid Sphere about a Diameter

Let M and R be the mass and radius of the given solid sphere. Let O be the centre and OX the given axis. Draw two spheres of radii x and $x+dx$ concentric with the given solid sphere. The thin spherical shell trapped between these spheres may be treated as a hollow sphere of radius x. The mass per unit volume of the solid sphere

$$= \frac{M}{\frac{4}{3}\pi R^3} = \frac{3M}{4\pi R^3}$$

The thin hollow sphere considered above has a surface area $4\pi x^2$ and thickness dx. Its volume is $4\pi x^2 dx$ and hence its mass is

$$= \left(\frac{3M}{4\pi R^2}\right)\left(4\pi x^2 dx\right) = \frac{3M}{R^2}x^2 dx$$



Its moment of inertia about the diameter OX is, therefore,

$$dI = \frac{2}{3}\left[\frac{3M}{R^3}x^2 dx\right]x^2 = \frac{2M}{R^3}x^4 dx$$

If $x = 0$, the shell is formed at the centre of the solid sphere. As x increases from 0 to R, cover the whole solid sphere.

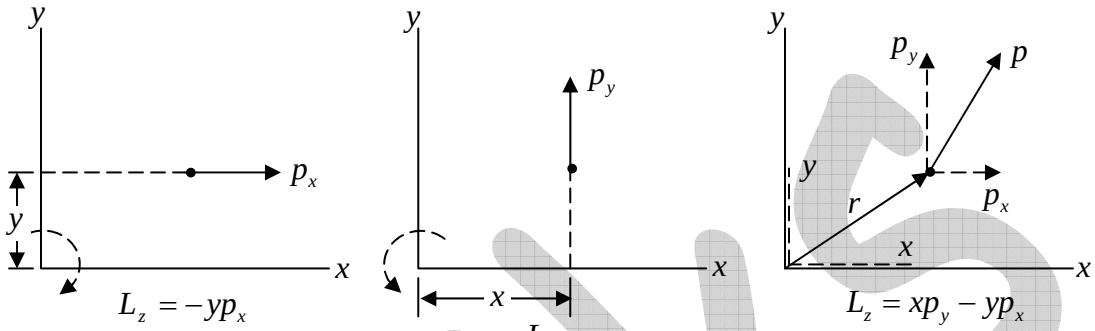
The moment of inertia of the solid sphere about OX is, therefore,

$$I = \int_0^R \frac{2M}{R^3}x^4 dx = \frac{2}{5}MR^2.$$

4.3 Rotational Dynamics

4.3.1Angular Momentum

Consider motion in the xy plane, first in the x direction and then in the y direction, as in drawings a and b on the next page.



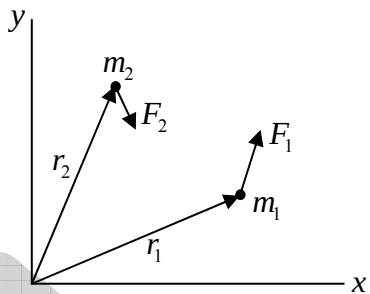
The most general case involves both these motions simultaneously, as drawings above show. Hence $L_x = xp_y - yp_x$ as you can verify by inspection or by evaluating the cross product as follows. Using $r = (x, y, 0)$ and $p = (p_x, p_y, 0)$, we have

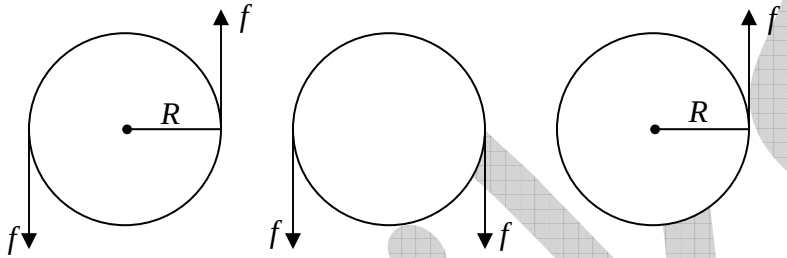
$$L = r \times p = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & 0 \\ p_x & p_y & 0 \end{vmatrix} = (xp_y - yp_x)\hat{k}$$

We have limited our illustrations to motion in the xy plane where the angular momentum lies entirely along the z axis. There is however, no difficulty applying any of these methods to the general case where L components along all three has axes.

4.3.2 Torque

The torque $\vec{\tau} = \vec{r} \times \vec{F}$ that $\vec{\tau}$ and $\vec{F}$ are always perpendicular. There can be a torque on a system with zero net force, and there can be force with zero net torque. In general, there will be both torque and force. These three cases are illustrated in the sketches below. (The torques are evaluated about the centers of the disks.)





$\tau = 2Rf$, $F = 0$ | $\tau = 0$, $F = 2f$ | $\tau = Rf$, $F = f$

Torque is important because it is intimately related to the rate of change of angular momentum:

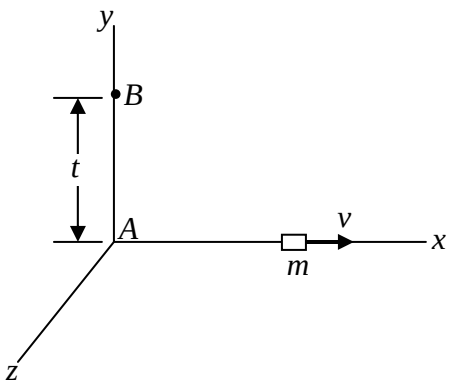
$$\frac{d\vec{L}}{dt} = \frac{d}{dt}\left(\vec{r} \times \vec{p}\right) = \left(\frac{d\vec{r}}{dt} \times \vec{p}\right) + \left(\vec{r} \times \frac{d\vec{p}}{dt}\right)$$

But $\left(d\vec{r}/dt\right) \times \vec{p} = \vec{v} \times m\vec{v} = 0$, since the cross product of two parallel vectors is zero. Also, $d\vec{p}/dt = \vec{F}$, by Newton's second law. Hence, the second terms is $\vec{r} \times \vec{F} = \vec{\tau}$, and we have

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

shows that if the torque is zero, $\vec{L} =$ constant and the angular momentum is conserved.

Example: Consider a block of mass m and negligible dimensions sliding freely in the x direction with velocity $\vec{v} = v\hat{i}$ as shown in the sketch.

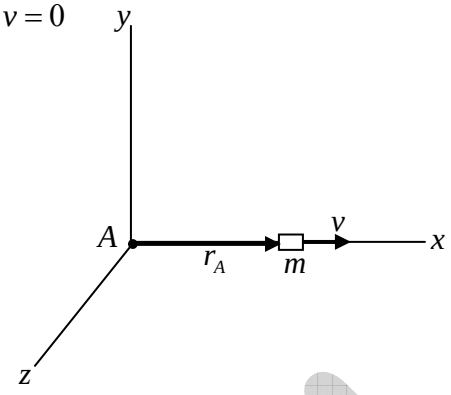


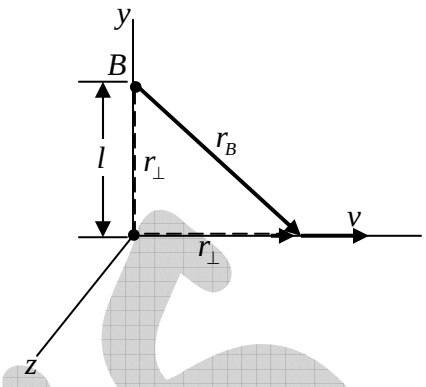
(a) What is its angular momentum L_A about origin A and its angular momentum L_B about the origin B?

(b) What is its torque L_A about origin A and its angular momentum L_B about the origin B?

Solution: As shown in the figure given below the vector from origin A to the block is $r_A = x\hat{i}$.

Since $r_A = mr_A \times v = 0$





Taking origin B, we can resolve the position vector r_B into a component r_l parallel to v and a component $r_\perp$ perpendicular to v. Since $r_\parallel \times v = 0$, only $r_\perp$ gives a contribution to L_B. We have $|r_\perp \times v| = lv$ and

$$L_B = mr_B \times v = mlv\hat{k}$$

L_B lies in the positive z direction because the sense of rotation is counter-clockwise about the z - axis.

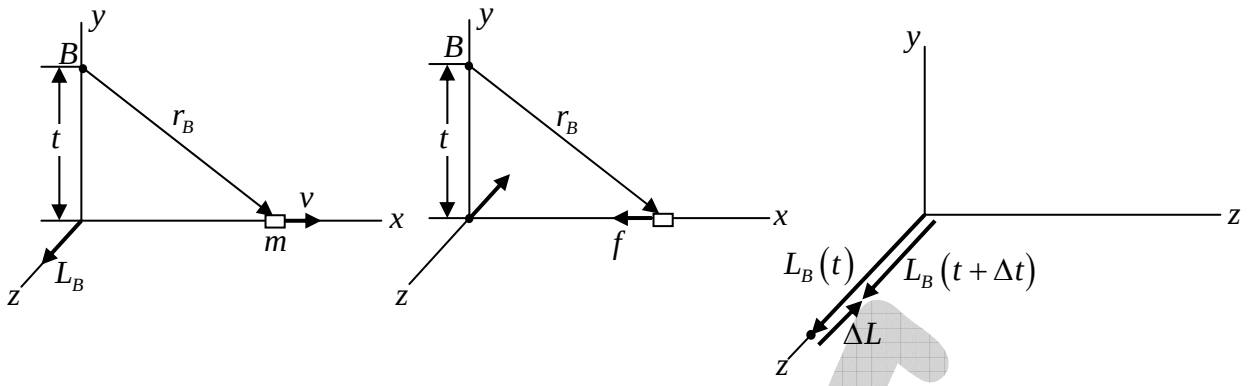
To calculate L_B formally we can write $r_B = x\hat{i} - l\hat{j}$ and evaluate $r_B \times v$ using our determinantal form.

$$L_B = mr_B \times v = m\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & -l & 0 \\ v & 0 & 0 \end{vmatrix} = mlv\hat{k} \text{ as before.}$$

The following example shows in a striking way how L depends on our choice of origin.

b) For a simple illustration of the relation $\tau = dL/dt$, consider a small block of mass m sliding in the z direction with velocity $v = v\hat{i}$. The angular momentum of the block about origin B is

$$L_B = mr_B \times v = mlv\hat{k} \quad \text{.....(1)}$$



as we discussed in **problem (a)** . If the block is sliding freely, v does not change, and L_B is therefore constant, as we expect, since there is no torque acting on the block.

Suppose now that the block slows down because of a friction force $f = -f\hat{i}$. The torque on the block about origin B is

$$\tau_B = r_B \times f = -lf\hat{k} \qquad (2)$$

We see from Eq. (1) that as the block slows, L_B remains along the positive z direction but its magnitude decreases. Therefore, the change ΔL_B in L_B points in the negative z direction, as shown in the lower sketch. The direction of ΔL_B is the same as the direction of τ_B. Since $\tau_B = dL/dt$ in general, the vectors τ and ΔL are always parallel.

From Eq. (1), $\Delta L_B = ml\Delta v\hat{k}$ (3)

Where $\Delta v < 0$. Dividing equation (3) by Δt and taking the limit $\Delta t \to 0$, we have

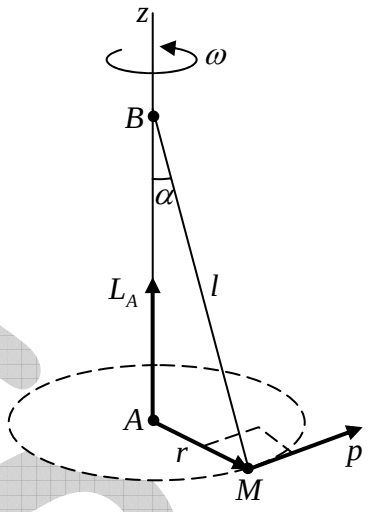
$$\frac{dL_B}{dt} = ml\frac{dv}{dt}\hat{k} \qquad (4)$$

By Newton's second law, $m\,dv/dt = -f$ and equation (4) becomes

$\frac{dL_B}{dt} = -lf\hat{k} = \tau_B$, as we expect.

It is important to keep in mind that since τ and L depend on the choice of origin, the same origin must be used for both when applying the relation $\tau = dL/dt$, as we were careful to do in this problem. The angular momentum of the block in this example changed only in magnitude and not in direction, since τ and L happened to be along the same line

Example: Assume that the Conical pendulum is in steady circular motion with constant angular velocity ω.then discuss angular momentum and Torque, for the Conical Pendulum about point A and point B



Solution: To calculate of angular momentum

We begin by evaluating L_A, the angular momentum about origin A.

From the sketch we see that L_A lies in the positive z direction. It has magnitude $|r_\perp||p| = |r|\,|p| = rp$, where r is the radius of the circular motion. Since $|p| = Mv = Mr\omega$ we have $L_A = Mr^2\omega k$

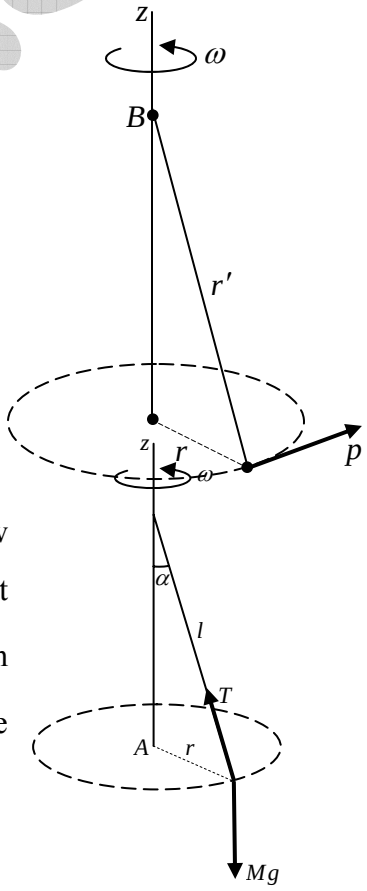
Note that L_A is constant, both in magnitude and direction.

Now let us evaluate the angular momentum about the origin B located at the pivot. The magnitude of L_B is

$$|L_B| = |r' \times p| = |r'|\,|p| = l|p| = Mlr\omega$$

where $|r'| = l$ the length of the string. It is apparent that the magnitude of L depends on the origin we choose.

Unlike L_A, the direction of L_B is not constant. L_B is perpendicular to both r' and p, and the sketches below show L_B at different times. Two sketches are given to emphasize that only the magnitude and direction of L are important, not the position at which we choose to draw it. The magnitude of L_B is constant, but its direction is obviously not constant as the bob swings around, L_B sweeps out the shaded cone shown in the sketch at the right. The z component of L_B is constant, but the horizontal component travels around the circle with the bob.



Calculation of Torque

we calculated the **torque** of a conical pendulum about two different origins. Now we shall complete the analysis by showing that the relation $\tau = dL/dt$ is satisfied.

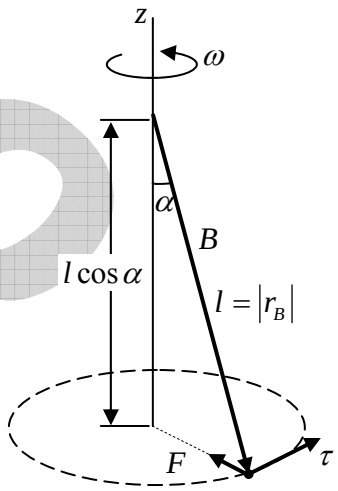
The sketch illustrates the forces on the bob. T is the tension in the string. For uniform circular motion there is no vertical acceleration, and consequently

$$T\cos\alpha - Mg = 0 . \qquad (1)$$

The total force F on the bob is radially inward: $F = -T\sin\alpha\hat{r}$.

The torque on M about A is $\tau_A = r_A \times F = 0$,

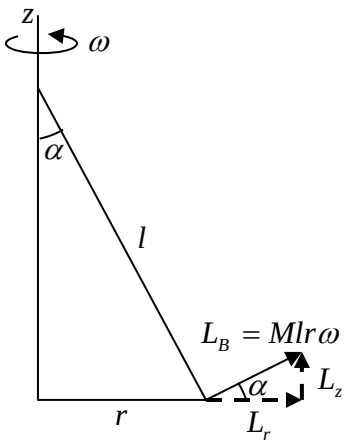
since r_A and F are both in the $\hat{r}$ direction. Hence $\frac{dL_A}{dt} = 0$. and we have the result $L_A = \text{constant}$, as we already know.



The problem looks entirely different if we take the origin at B. The torque τ_B is $\tau_B = r_B \times F$.

Hence, $|\tau_B| = l\cos\alpha F = l\cos\alpha F = l\cos\alpha T\sin\theta = Mgl\sin\alpha$

where we have used Eq. (1). $T\cos\alpha = Mg$. The direction of τ_B is tangential to the line of motion of M .



$$\tau_B = Mgl\sin\alpha\hat{\theta}$$

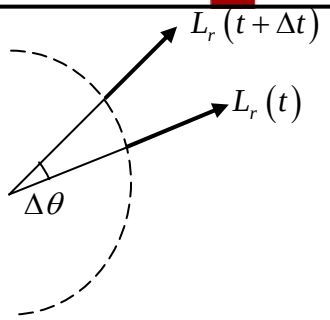
where $\hat{\theta}$ is the unit tangential vector in the plane of motion.

Our problem is to show that the relation

$$\tau = \frac{dL_B}{dt}$$

is satisfied. we know that L_B has constant magnitude $Mlr\omega$. As the diagram at left shows, L_B has a vertical component $L_z = Mlr\omega\sin\alpha$ and a horizontal radial component $L_r = Mlr\omega\cos\alpha$. Writing $\vec{L}_B = \vec{L}_z + \vec{L}_r$, we see that L_z is constant; it changes direction as the bob swings around. However, the magnitude of L_r is constant. We encountered such a situation in section 1.8, where we showed that the only way a vector A of constant

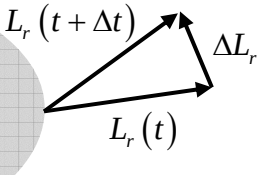
magnitude can change in time is to rotate and that if its instantaneous rate of rotation is $d\theta/dt$, then $|dA/dt| = Ad\theta/dt$. We can employ this relation directly to obtain



$$\left|\frac{dL}{dt}\right| = L_r\omega .$$

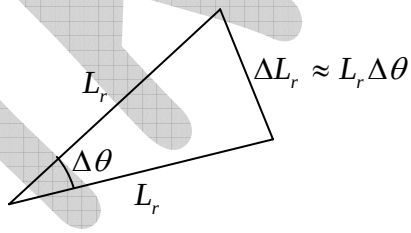
However, since we shall invoke this result frequently, let us take a moment to rederive it geometrically.

The vector diagrams show L at some time t and at $t+\Delta t$ During the interval Δt, the bob swings through angle $\Delta\theta = \omega\Delta t$, and L, rotate through the same angle. The magnitude of the vector difference $\Delta L_r = L_r(t+\Delta t) - L_r(t)$ is given approximately by



$$|\Delta L_r| \approx L_r\Delta\theta$$

In the limit $\Delta t \to 0$, we have



$$\frac{dL_r}{dt} = L_r\frac{d\theta}{dt} = L_r\omega$$

Since $L_r = Mlr\omega\cos\alpha$, $\frac{dL_r}{dt} = Mlr\omega^2\cos\alpha$.

$Mr\omega^2$ is the radial force, $T\sin\alpha = Mg$, we have $\frac{dL_r}{dt} = Mgl\sin\alpha$

which agrees with tie magnitude of τ_B front Eq. (2), Furthermore, as the vector drawings indicate, $\frac{dL_r}{dt}$ lies in the tangential direction. Parallel to τ_B, as we expect.

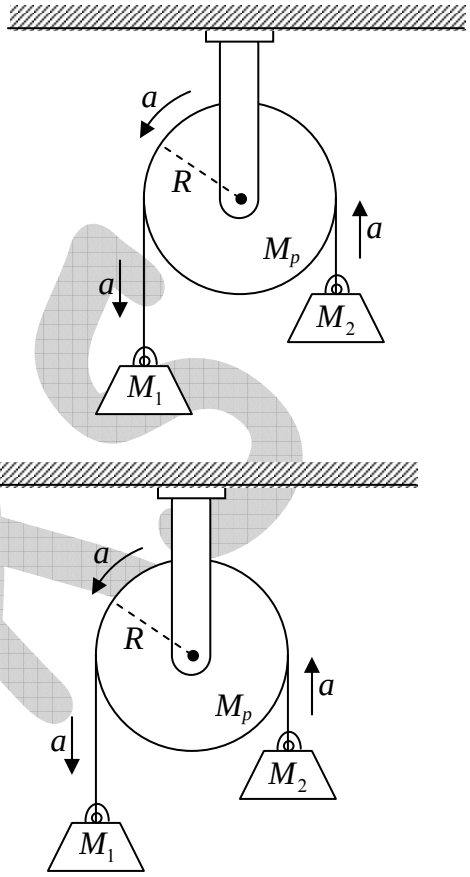
Another way to calculate $\frac{dL_B}{dt}$ is to write L_B in vector form and then differentiate:

$$\vec{L}_B = (Mlr\omega\sin\alpha)\hat{k} + (Mlr\omega\cos\alpha)\hat{r}$$

$$\frac{dL_B}{dt} = Mlr\omega\cos\alpha\frac{d\hat{r}}{dt} = Mlr\omega^2\cos\alpha\hat{\theta} \qquad \text{where we have used } \frac{d\hat{r}}{dt} = \omega\hat{\theta} .$$

Example: Atwood's Machine with a Massive Pulley

The problem is to find the acceleration a for the arrangement shown in the sketch. The effect of the pulley is to be included. The masses of particles are M_1, M_2 and mass of pulley is M_P .The radius of pulley R .



Solution: Force diagrams for the three masses are shown below. The points of application of the forces on the pulley are shown; this is necessary whenever we need to calculate torques. The pulley evidently goes pure rotation about its axis, so we take the axis of rotation to be the axle.

The equations of motion are

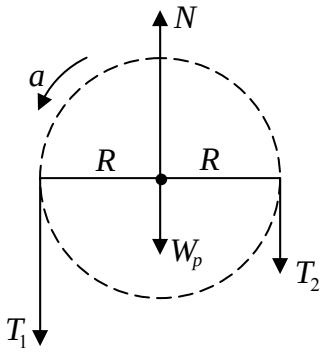
$$W_1 - T_1 = M_1 a$$
$$T_2 - W_2 = M_2 a$$

Masses

$$\gamma = T_1 R - T_2 R = I\alpha$$
$$N - T_1 - T_2 - W_p = 0$$

Pulley

Note that in the torque equation, a must be positive counterclockwise to correspond to our convention that torque out of the paper is positive. N is the force on the axle, and the last equation simply assures that the pulley does not fall. Since we don't need to know N it does not contribute to the solution.



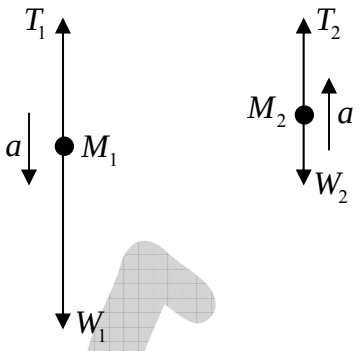
There is a constraint relating a and α , assuming that the rope does not slip. The velocity of the rope is the velocity of a point on the surface of the wheel, $v = \omega R$, from which it follows that $a = \alpha R$.

We can now eliminate T_1, T_2 and α;

$$W_1 - W_2 - (T_1 - T_2) = (M_1 + M_2)a$$

$$T_1 - T_2 = \frac{I\alpha}{R} = \frac{Ia}{R^2}$$

$$W_1 - W_2 - \frac{Ia}{R^2} = (M_1 + M_2)a$$



If the pulley is a simple disk, we have

$$I = \frac{M_p R^2}{2}$$

and it follows that

$$a = \frac{(M_1 - M_2)g}{M_1 + M_2 + \frac{M_p}{2}}.$$

The Pulley increases the total inertial mass of the system, but in comparison with the hanging weights, the effective mass of the pulley is only one-half its real mass.

4.4 Motion Involving Both Translation and Rotation

By using center of mass coordinates we will find it a straightforward matter to obtain simple expressions for both the angular momentum and the torque and to find the dynamical equation connecting them.

As before, we shall consider only motion for which the axis of rotation remains parallel to the z-axis. We shall show that L_z the z component of the angular momentum of the body, can be written as the sum of two terms. L_z is the angular momentum $I_0\omega$ due to rotation of the body about its center of mass, plus the angular momentum $(R \times MV)_z$ due to motion of the center of mass with respect to the origin of the inertial coordinate system:

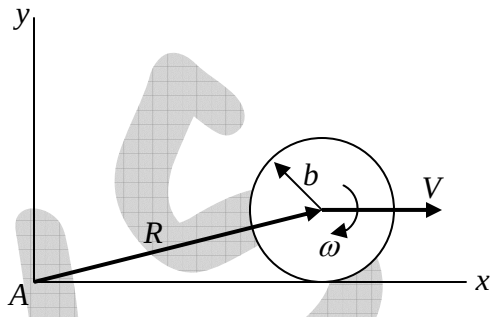
$$L_z = I_0\omega + (R \times MV)_z$$

where R is the position vector of the center of mass and $V = \dot{R}$.

4.4.1 Angular Momentum of a Rolling Wheel

In this example we apply to the calculation of the angular momentum of a uniform wheel of mass M and radius b which rolls uniformly and without slipping. The moment of inertia of the wheel about its center of mass is $I_0 = \frac{1}{2}Mb^2$ and its angular momentum about the center of mass is

$$L_0 = -I_0\omega = -\frac{1}{2}Mb^2\omega .$$



L_0 is parallel to the axis. The minus sign indicates that L_0 is directed into the paper in the negative z direction.

If we calculate the angular momentum of the center of mass of the wheel with respect to the origin, we have

$$\left(R \times MV\right)_z = -MbV$$

The total angular momentum about the origin is then

$$L_z = -\frac{1}{2}Mb^2\omega - MbV = -\frac{1}{2}Mb^2\omega = -\frac{1}{2}Mb^2\omega$$

where we have used the result $V = b\omega$, which holds for a wheel that rolls without slipping.

Torque also naturally divides itself into two components.

4.4.2 The Torque on a Body

Fixed axis rotation $\vec{\omega} = \omega\hat{k}$.

If τ_0 is the component of the torque about the center of mass and $F = \Sigma f_j$ is the total applied force

Then torque is given by $\tau_z = \tau_0 + \left(R \times F\right)_z$

The first term in is the torque about the center of mass due to the various external forces, and the second term is the torque due to the total external force acting at the center of mass.

Example: A disk of mass M and radius b is pulled with constant force F by a thin tape wound around its circumference. The disk slides on ice without friction .what will be torque and acceleration of the system .

We choose a coordinate system whose origin A is along the line of F. The torque about A is, from

$$\tau_z = \tau_0 + \left(R \times F\right)_z = bF - bF = 0$$

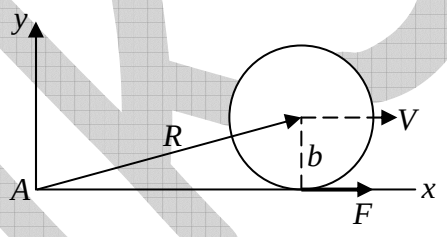
The torque is zero. As we expect, and angular momentum about the origin is conserved. The angular momentum about A.

$$L_z = I_0\omega + \left(R \times MV\right)_z = I_0\omega - bMV .$$

Since $\frac{dL_z}{dt} = 0$, we have

$$0 = I_0\alpha - bMa$$

or $$\alpha = \frac{bMa}{I_0} = \frac{bF}{I_0}, \text{ as before.}$$



4.4.3 Condition of Rolling Without Slipping

If velocity of center of mass is V and ω is angular velocity of center of mass then $V = \omega R$ where R is radius of rolling body .similarly if acceleration of center of mass is a and α is angular velocity of about center of mass then $a = \alpha R$ where R is radius of rolling body.

Example -Drum Rolling Down a Plane

A uniform drum of radius b and mass M rolls a without slipping down a plane inclined at angle θ. Find its velocity along the plane. The moment of inertia of the drum about its axis is $I_0 = \frac{Mb^2}{2}$. Then find the speed at the lowest end of inclined plane

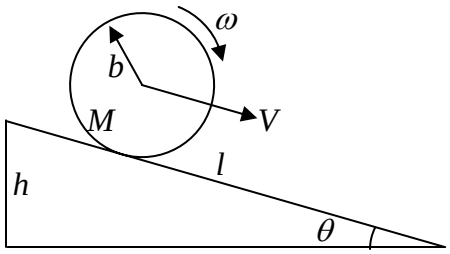
where $l = \frac{h}{\sin\theta}$ is the displacement of the center of mass as the drum descends height h.

Change in potential energy is equal to gain in kinetic energy

So $Mgh = \frac{1}{2}\left(M\right)V^2 + \frac{1}{2}\left(I_0\right)\omega^2$

For Rolling without slipping $V = \omega R$ and $I_0 = \frac{Mb^2}{2}$

$$Mgh = \frac{1}{2}\left(M + \frac{M}{2}\right)V^2 = \frac{3}{4}MV^2 \quad \text{or} \quad V = \sqrt{\frac{4gh}{3}}.$$



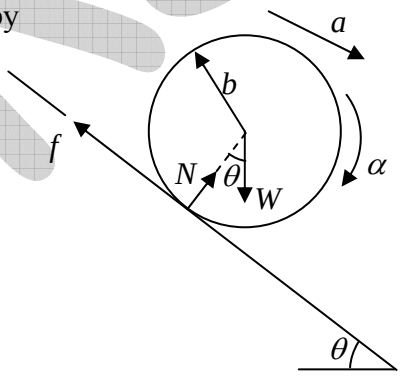
Example: A uniform drum of radius b and mass M rolls a without slipping down a plane inclined at angle θ. Find its acceleration along the plane. The moment of inertia of the drum about its axis is $I_0 = \frac{Mb^2}{2}$.

Solution: The forces acting on the drum are shown in the diagram. f is the force of friction. The translation of the center of mass along the plane is given by

$$W\sin\theta - f = Ma$$

and the rotation about the center of mass by

$$bf = I_0\alpha.$$



For rolling without slipping, we also have

$$a = b\alpha.$$

If we eliminate f, we obtain

$$W\sin\theta - I_0\frac{\alpha}{b} = Ma$$

Using $I_0 = \frac{Mb^2}{2}$, and $\alpha = \frac{a}{b}$, we obtain

$$Mg\sin\theta - \frac{Ma}{2} = Ma,$$

or $$a = \frac{2}{3}g\sin\theta.$$

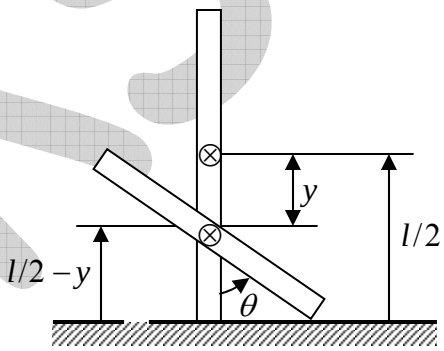
Example: A stick of length l and mass M initially upright on a frictionless table, starts falling. The problem is to find the speed of the center of mass as a function of position.

The key lies in realizing that since there are no horizontal forces, the center of mass must fall straight down. Since we must find velocity as a function of position, it is natural to apply energy methods.

The sketch shows the stick after it has rotated through angle θ and the center of mass has fallen distance y. The filial energy is $E = K_0 + U_0 = \dfrac{Mgl}{2}$

The kinetic energy at a later time is $K = \dfrac{1}{2} I_0 \dot{\theta}^2 + \dfrac{1}{2} M \dot{y}^2$

and the corresponding potential energy is $U = Mg\left(\dfrac{l}{2} - y\right)$.



Since there are no dissipative forces, mechanical energy is conserved and $K + U = K_0 + U_0 = \dfrac{Mgl}{2}$. Hence

$$\frac{1}{2} M\dot{y}^2 + \frac{1}{2} I_0 \dot{\theta}^2 + Mg\left(\frac{l}{2} - y\right) = Mg\frac{l}{2}.$$

We can eliminate θ by turning to the constraint equation. From the sketch we see that

$$y = \frac{l}{2}(1 - \cos\theta).$$

Hence, $\dot{y} = \dfrac{l}{2}\sin\theta\dot{\theta}$ and $\dot{\theta} = \dfrac{2}{l\sin\theta}\dot{y}$

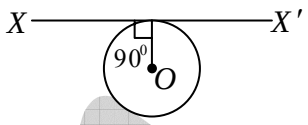
Since $I_0 = M\left(\dfrac{l^2}{12}\right)$, we obtain

$$\frac{1}{2} M\dot{y}^2 + \frac{1}{2} M \frac{l^2}{12}\left(\frac{2}{l\sin\theta}\right)\dot{y}^2 + Mg\left(\frac{l}{2} - y\right) = Mg\frac{l}{2}$$

or $$\dot{y}^2 = \frac{2gy}{\left[1 + \dfrac{1}{(3\sin^2\theta)}\right]} \Rightarrow \dot{y} = \left[\frac{6gy\sin^2\theta}{3\sin^2\theta + 1}\right]^{1/2}$$

MCQ (Multiple Choice Questions)

Q1. A thin wire of length L and uniform linear mass density ρ is bent into a circular loop with centre of O as shown. The moment of inertia of the loop about the axis XX' is



(a) $\frac{\rho L^3}{8\pi^2}$ (b) $\frac{\rho L^3}{16\pi^2}$

(c) $\frac{5\rho L^3}{16\pi^2}$ (d) $\frac{3\rho L^3}{8\pi^2}$

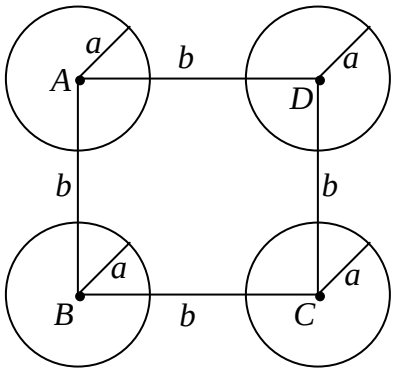
Q2. One quarter sector is cut from a uniform circular disc of radius R. This sector has a mass M. If is made to rotate about a line perpendicular to its plane and passing through the centre of the original disc. Its moment of inertia about the axis of rotational is

(a) $\frac{1}{2}MR^2$ (b) $\frac{1}{4}MR^2$ (c) $\frac{1}{8}MR^2$ (d) $\sqrt{2}MR^2$

Q3. From a circular disc of radius R and mass $9M$ a small disc of radius $\frac{R}{3}$ is removed from the disc. The moment of inertia of the remaining disc about an axis perpendicular to the plane of the disc and passing through O is

(a) $4MR^2$ (b) $\frac{40}{9}MR^2$ (c) $10\,MR^2$ (d) $\frac{37}{9}MR^2$

Q4. Four spheres of diameter $2a$ and mass M each are placed with their centres on the four corners of a square of side b. The moment of inertia of the system about one side of the square taken as the axis is



(a) $\frac{8M\left(3a^2+2b^2\right)}{3}$ (b) $\frac{9M\left(6a^2+4b^2\right)}{8}$

(c) $\frac{3M\left(2a^2+4b^2\right)}{7}$ (d) $\frac{2M\left(4a^2+5b^2\right)}{5}$

Q5. The ratio of radius of gyration of a circular ring and a disc of the same radius about the axis passing through their centres and perpendicular to their plane

(a) $\sqrt{5}:1$ (b) $\sqrt{2}:1$ (c) $1:\sqrt{3}$ (d) $1:\sqrt{2}$

Q6. The moment of inertia of a thin rod of length L and M about an axis that is perpendicular to the rod and of a distance x from its centre is

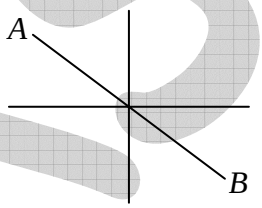
(a) $\frac{ML^2}{12}+Mx^2$ (b) $\frac{ML^2}{6}+Mx^2$

(c) $\frac{ML^2}{3}+\frac{Mx^2}{7}$ (d) $\frac{ML^2}{12}+Mx^2$

Q7. Two uniform identical rods each of mass M and length l are joined to form a cross as shown in the figure. The moment of inertia of the cross about the bisector AB is

(a) $\frac{Ml^2}{4}$ (b) $\frac{Ml^2}{5}$

(c) $\frac{Ml^2}{8}$ (d) $\frac{Ml^2}{12}$



Q8. The moment of inertia of a pair spheres, each having a mass m and radius r, kept in contact about the tangent passing through the point of contact is

(a) $\frac{7mr^2}{5}$ (b) $\frac{9mr^2}{5}$ (c) $\frac{14mr^2}{5}$ (d) $\frac{18mr^2}{5}$

Q9. From a given sample of uniform wire two circular loops P and Q are made, P of radius r and Q of radius nr. If the M.I. of Q about its axis is four times that of P about its axis, the value of n is

(a) $4^{\frac{2}{3}}$ (b) $4^{\frac{1}{3}}$ (c) $4^{\frac{1}{2}}$ (d) $4^{\frac{1}{4}}$

Q10. Two circular discs A and B are of equal masses and thickness but made of metal with densities d_A and $d_B\left(d_A > d_B\right)$. If their moment of inertia about an axis passing through their centres and perpendicular to their faces are I_A and I_B, then

(a) $I_A = I_B$ (b) $I_A > I_B$

(c) $I_A < I_B$ (d) $I_A \geq I_B$

Q11. The angular momentum of a particle rotating under a central force is constant due to

(a) constant force (b) constant linear momentum

(c) constant torque (d) zero torque

Q12. A solid sphere is rotating in free space. If the radius of the sphere is increased keeping mass same which are of the following will not be affected:

(a) moment of inertia (b) angular momentum

(c) angular velocity (d) rotational kinetic energy

Q13. A thin horizontal circular disc is rotating about a vertical axis passing through its centre. An insect is at rest at a point near the rim of the disc. The insect now moves along a diameter of the disc to react its outer end. During the journey of the insect, the angular speed of the disc

(a) remains unchanged (b) continuously decreases

(c) continuously increases (d) first increases and then decreases

Q14. A couple is acting on a two-particle system connnected by a massless rod. The resultant motion will be

(a) purely rotational motion (b) purely translational motion

(c) both (a) and (b) (d) Neither (a) nor (b)

Q15. Angular momentum of a system of particles is conserved

(a) when no external force acts on the system

(b) when no external torque acts on the system

(c) when no external impulse acts on the system

(d) when axis or rotation remains the same

Q16. The direction of angular velocity vector is along

(a) the tangent to the circular path (b) the inward radius

(c) the outward radius (d) the axis of rotation

Q17. A body is projected from the ground with some angle to the horizontal. What happens to its angular momentum about the initial position in its motion.

(a) decreases (b) increases

(c) remains the same (d) first increases then decreases

Q18. A disc is rolling, the velocity of its centre of mass is v_{cm}. Which one will be correct?

(a) the velocity of highest point is $2v_{cm}$ and the point of contact is zero.

(b) the velocity of highest point is v_{cm} and the point of contact is v_{cm}

(c) the velocity of highest point is $2v_{cm}$ and point of contact is v_{cm}.

(d) the velocity of highest point is $2v_{cm}$ and point of contact is $2v_{cm}$

Q19. For a hollow cylinder and a solid cylinder rolling without slipping on an incline plane, which of them will reach earlier

(a) solid cylinder (b) hollow cylinder

(c) both simultaneously (d) can't say anything

Q20. Pick out the correct statement about the torque vector

(a) The torque acting on a particle is given by $\vec{F}\times\vec{r}$ where $\vec{r}$ is the position vector of the particle about the point chosen for calculating torque.

(b) If net torque acting on a particle is zero, its angular momentum is not constant

(c) Zero net torque acting on a system implies zero net force

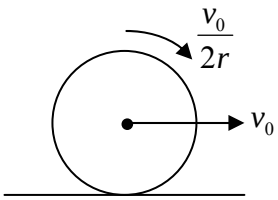
(d) If the angular momentum of a system changes with time, a net torque must act on the system

Q21. A cylinder is released from rest from the top of an incline of inclination θ and length l. If the cylinder rolls without slipping, what will be its speed when it reaches the bottom?

(a) $\sqrt{\frac{4gl\sin\theta}{3}}$ (b) $\sqrt{\frac{2gl\sin\theta}{3}}$

(c) $\sqrt{\frac{5gl\sin\theta}{3}}$ (d) $\sqrt{\frac{6gl\sin\theta}{3}}$

Q22. A sphere of mass M and radius r shown in the figure slips on a rough horizontal surface. At some time it has translational velocity v_0 and rotational velocity about the centre of mass $\frac{v_0}{2r}$. When the sphere starts pure rolling the transitional velocity is



(a) $\frac{2v_0}{7}$ (b) $\frac{3v_0}{9}$ (c) $\frac{4v_0}{9}$ (d) $\frac{6v_0}{7}$

Q23. The centre of a wheel rolling on a plane surface moves with speed v_0. A particle on the rim of the wheel at the same level as the centre will be moving at speed:

(a) zero (b) v_0 (c) $\sqrt{2}v_0$ (d) $2v_0$

Q24. Two small kids weighing $10\,kg$ and $15\,kg$ are trying to balance a seesaw of total length $5.0\,m$. If the kid of mass $10\,kg$ sits at an end at what distance from the centre the other kid should sit?

(a) $1.2\,m$ (b) $1.5\,m$ (c) $1.3\,m$ (d) $1.7\,m$

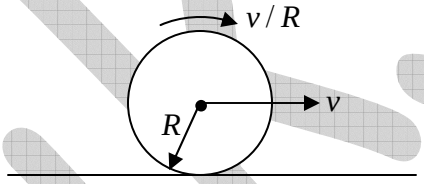
Q25. A disc is performing pure rolling on a smooth stationary surface with constant angular velocity as shown in figure. At any instant for the lowermost point of the disc.

(a) velocity is v, acceleration is zero

(b) velocity is zero, acceleration is zero

(c) velocity is v, acceleration is $\frac{v^2}{R}$

(d) velocity is zero, acceleration is $\frac{v^2}{R}$

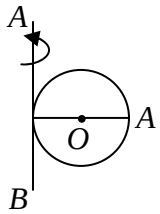


NAT (Numerical Answer type)

Q26. Three rings each of mass $10\,kg$ and radius $2\,m$ are placed such that they touch each other. The moment of inertia of about the axis shown is _____ $kg\cdot m^2$.

Q27. The moment of inertia of a solid cylinder of mass $10\,kg$ and radius $3\,m$ about a line parallel to axis of the cylinder and on the surface of the disc is _____ $kg\cdot m^2$.

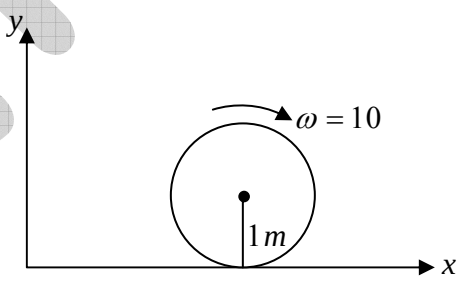
Q28. A disc of mass $8\,kg$ and radius $2\,m$ is rotating about the axis AB, that is tangent to the disc. If the linear speed of point A on the periphery of the disc is $20\,m/s$, then the kinetic energy of the disc is _____ T.



Q29. Three point masses $m_1 = 4\,kg, m_2 = 3\,kg$ and $m_3 = 5\,kg$ are located at the vertices of an equilateral triangle of side length $5\,m$. The M.I. of the system about the median passing through mass m_1 is _____ $kg\cdot m^2$.

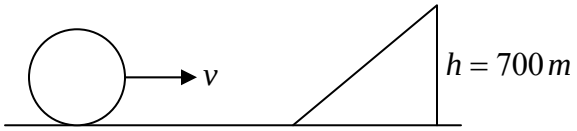
Q30. The M.I. of a uniform square plate of side $4\,m$ and mass $6\,kg$ about an axis perpendicular to its plane and passing through one of its corners is _____ $kg\cdot m^2$.

Q31. A disc of mass $2\,kg$ and radius $1\,m$ is rolling with angular speed $10\ rad/s$ on a horizontal plane as shown in figure. The magnitude of angular momentum of the disc about the origin is _____.



Q32. A uniform body of radius $10\,m$, mass $5\,kg$ and moment of inertia $2\,kg\cdot m^2$ rolls down an incline plane making an angle 37^0 with the horizontal. [Take $\sin 37^0 = 3/5$, $\cos 37^0 = 4/5$ and $g = 10\,m/s^2$]. Then the acceleration is _____ m/s^2.

Q33. A solid sphere is rolling on a frictionless surface with a translational velocity v (in ms^{-1}) as shown in the figure. If it is to climb the incline plane, then v should be _____ m/s.



Q34. A uniform cube of side $4\,m$ and mass $30\,kg$ rests on a rough horizontal plane. A horizontal force F is applied normal to one of the faces at a point that is directly above the centre of the face at a height $3\,m$ above the box. The minimum value of F for which the cube begins to tip above the edge (assuming no sliding) is_____. (Take $g = 10\,m/s^2$).

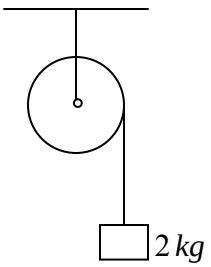
Q35. A stone of mass m, tied to the end of a spring, is whirled around a horizontal circle. The length of the string is gradually reduced keeping the angular momentum of the stone about the centre constant. Then the tension in the string is given by $T = Ar^n$, where A is a constant, r, the instantaneous radius of the circle. Then n is _____.

Q36. A rod of weight $20\,N$ is supported by two parallel knife edges A and B and is in equilibrium in a horizontal position. The knives are at a distance $30\,m$ from each other. The centre of mass of the rod is at a distance of $10\,m$ from A. The normal reaction on A is _____ N.

Q37. A thin uniform circular disc of mass M and radius R is rotating in a horizontal plane about an axis passing through its centre and perpendicular to its plane with an angular velocity 10 rad/s. Another disc of same dimensions but of mass $\frac{M}{4}$ is placed gently on the first disc coaxially. The angular velocity of the system now is _____ rad/s.

Q38. A rod of length $1\,m$ and mass $2\,kg$ is hinged at point O. A small bullet of mass $1\,kg$ hits the rod as shown in the figure. The bullet gets embedded in the rod. Angular velocity of the system just after impact, is _____ rad/s^2.

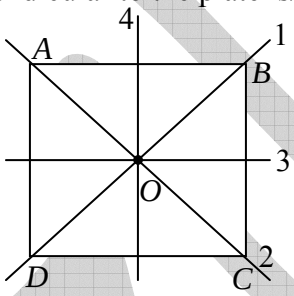
Q39. A block of mass $2\,kg$ is attached to the end of an inextensible string which is wound over a rough pulley of mass $6\,kg$. Assume the string does not slide over the pulley. The acceleration of the block when released is _____ m/s^2.



Q40. A block of mass $150\,kg$ is attached to a chord passing through a hole in a horizontal frictionless surface. The block is initially revolving in a circle of radius is $0.5\,m$ about the hole, with tangential velocity of $4\,m/s$. The chord is then pulled slowly from below, shortening the radius of the circle in which the block revolves. The breaking strength of the chord is $600\,N$. The radius of the circle when the chord breaks is _____ m.

MSQ (Multiple Select Questions)

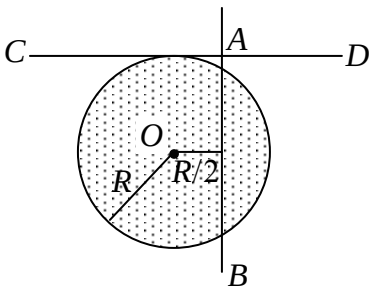
Q41. The moment of inertia of a thin square plate $ABCD$ of uniform thickness about an axis passing through the centre O and perpendicular to the plate is/are



(a) $I_1 + I_2$ (b) $I_3 + I_4$ (c) $I_1 + I_3$ (d) $I_1 + I_2 + I_3 + I_4$

(where $I_1, I_2\ I_3$ & I_4 are respectively the moment of inertia about axes $1, 2, 3$ & 4)

Q42. Consider a disk of mass M and radius R and the axes shown in the figure.



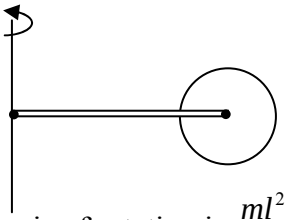
(a) The moment of inertia of the disc about an axis passing through the centre and perpendicular to the disc is $\frac{MR^2}{2}$.

(b) The moment of inertia of the disc about the line AB lying in the plane of the disc is $\frac{MR^2}{2}$.

(c) The moment of inertia of the disc about the line CD is $\frac{5}{4}MR^2$.

(d) The moment of inertia of the disc about the line CD is $\frac{3}{2}MR^2$

Q43. Consider a spherical ball of mass m and radius r. A thin rod of mass m and length l has one of its ends attached to the centre of the sphere and the other end touches the axis of rotation as shown in the figure.



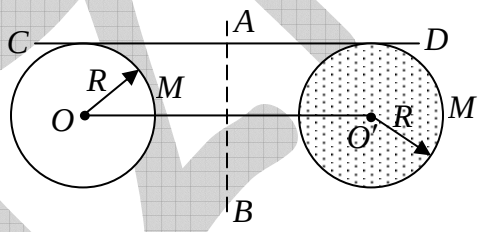
(a) The M.I. of the rod alone about the axis of rotation is $\frac{ml^2}{3}$

(b) The M.I. of the sphere alone about the axis of rotation is $\frac{2}{5}mr^2 + ml^2$

(c) The M.I. of the system about the axis of rotation is $\frac{ml^2}{3} + \left(\frac{2}{5}mr^2 + ml^2\right)$

(d) The M.I. of the system about the axis of rotation is $\frac{ml^2}{5} + \left(\frac{2}{7}mr^2 + 2ml^2\right)$

Q44. Consider a solid sphere and hollow sphere of same mass M and same radius R. Their centres are at a distance d from each other.



Line AB bisects OO' and line CD is tangent to both sphere then

(a) The moment of inertia of the system about the axis AB is $\frac{5}{7}MR^2 + Md^2$

(b) The moment of inertia of the system about the axis CD is $\frac{8}{16}MR^2 + 3MR^2$

(c) The moment of inertia of the system about the axis AB is $\frac{16}{15}MR^2 + \frac{Md^2}{2}$

(d) The moment of inertia of the system about the axis CD is $\frac{16}{15}MR^2 + 2MR^2$

Q45. Pick out the correct alternative(s)

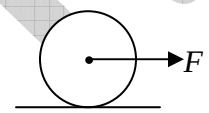
(a) The radius of gyration of a thin disc about any diameter is $\frac{R}{2}$, where R is the radius of the disc.

(b) The radius of gyration of a circular disc about a tangent in its plane is $\frac{\sqrt{5}}{2}R$, where R is the radius of the disc.

(c) The radius of gyration of a thin rod about an axis through its one end and perpendicular to the rod is $\frac{L}{\sqrt{3}}$, where L is the length of the rod.

(d) The radius of gyration of a rectangular lamina of sides l and b about an axis through its centre and perpendicular to its plane is $\frac{1}{2}\sqrt{\frac{l^2+b^2}{3}}$.

Q46. A solid sphere of mass M and radius R is pulled horizontally on a rough surface. Choose the incorrect alternative(s).



(a) The magnitude of frictional force is $\frac{F}{3}$

(b) The frictional force on the sphere acts forward

(c) The acceleration of the centre of mass is $\frac{2F}{3M}$

(d) The acceleration of centre of mass is $\frac{F}{M}$

Q47. Consider the uniform solid sphere, sphere (i) has radius r and mass m, sphere (ii) has radius r and mass $3m$, sphere (iii) has radius $3r$ and mass m. All the spheres are placed at the same point on the same inclined plane where they will roll without slipping to reach the bottom. If allowed to roll down the incline, then at the bottom of the incline

(a) sphere (i) will have the largest speed

(b) sphere (ii) will have the largest speed

(c) sphere (ii) will have the largest kinetic energy

(d) all the spheres will have equal speed

Q48. A rigid body is in pure rotation that is undergoing fixed axis rotation. Then which of the following statement(s) are true?

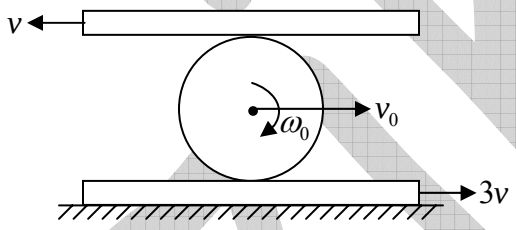
(a) You can find two points in the body in a plane perpendicular to the axis of rotation having the same velocity

(b) You can find two points in the body in a plane perpendicular to the axis of rotation having the same acceleration

(c) Speed of all particles lying on the curved surface of a cylinder whose axis coincides with the axis of rotation is the same.

(d) Angular speed of the body is the same as seen from any point in the body

Q49. The disc of radius r is confined to roll without slipping at A and B. If the plates have velocities as shown, then



(a) Angular velocity ω_0 of the disc is $\frac{2v}{r}$

(b) Linear velocity v_0 of the disc is $v_0 = v$

(c) Angular velocity ω_0 of the disc is $\frac{3v}{2r}$

(d) Linear velocity v_0 of the disc is $v_0 = 2v$

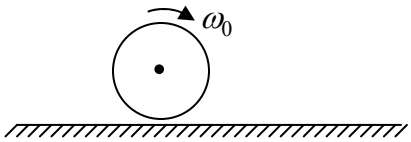
Q50. A disc is given an initial angular velocity ω_0 and placed gently on a rough horizontal surface as shown in the figure. The quantities which will depend on the coefficient of friction is/are

(a) The time until rolling begins

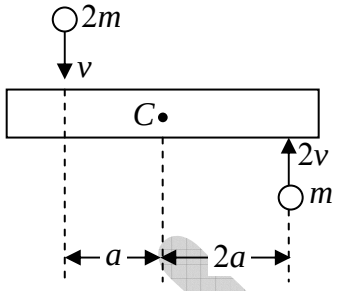
(b) The displacement of the disc until rolling begins

(c) The velocity when rolling begins

(d) The work done by force of friction



Q51. A uniform bar of length $6a$ and mass $8m$ lies on a smooth horizontal table. Two point masses m and $2m$ moving in the same horizontal plane with speed $2v$ and v respectively, strike the bas as shown and stick to the bas after collision. Denoting angular velocity, total energy and centre of mass velocity by ω, E and v_c respectively, we have



(a) $v_0 = 0$ (b) $\omega = \frac{3v}{5a}$ (c) $\omega = \frac{v}{5a}$ (d) $E = \frac{3mv^2}{5}$

Q52. The torque $\vec{\tau}$ on a body about a given by is found to be equal to $\vec{A}\times\vec{L}$ where $\vec{A}$ is a constant vector and $\vec{L}$ is the angular momentum of the body about the given point. From this it follows that

(a) $\frac{d\vec{L}}{dt}$ is perpendicular to $\vec{L}$ at all instant

(b) The component of $\vec{L}$ in the direction of $\vec{A}$ does not change with time

(c) The magnitude of $\vec{L}$ does not change with time

(d) $\vec{L}$ does not change with time

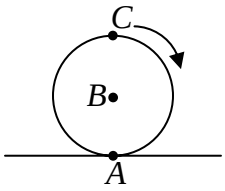
Q53. A sphere is rolling without slipping on a fixed horizontal plane surface. In the figure, A is the point of contact, B is the centre of the sphere and C is its topmost point. Then,

(a) $\vec{v}_C - \vec{v}_A = 2(\vec{v}_B - \vec{v}_C)$

(b) $\vec{v}_C - \vec{v}_B = \vec{v}_B - \vec{v}_A$

(c) $|\vec{v}_C - \vec{v}_A| = 2|\vec{v}_B - \vec{v}_C|$

(d) $|\vec{v}_C - \vec{v}_A| = 4|\vec{v}_B|$



Q54. A particle moves in a circular path with decreasing speed. Choose the correct alternative(s)

(a) Angular momentum remains constant

(b) Acceleration $\vec{a}$ is towards the centre

(c) Particle moves in a spiral path with decreasing radius

(d) The direction of angular momentum remains constant

Q55. For a rigid body rotating about a fixed axis

(a) The net torque acting on the body is always along the axis of rotation

(b) The angular velocity of all the particles have the same value

(c) The linear acceleration of all the particles have the same value

(d) The angular momentum and angular velocities always lie in the same direction

fiziks

Solutions

MCQ (Multiple Choice Questions)

Ans. 1: (c)

Solution: Mass of the ring $M = \rho L$

If R is the radius of the ring then $L = 2\pi R \Rightarrow R = \dfrac{L}{2\pi}$

M.I of the disc about an axis passing through O and parallel to XX' is

$I_0 = \dfrac{MR^2}{4}$

hence using parallel axis theorem, M.I about XX' is

$$I_{XX'} = I_0 + MR^2 = \frac{MR^2}{4} + MR^2 \Rightarrow I_{XX'} = \frac{5MR^2}{4} = \frac{5\rho L.\dfrac{L^2}{4\pi^2}}{4} \text{ or } I_{XX'} = \frac{5\rho L^3}{16\pi^2}$$

Ans. 2: (a)

Solution: The moment of inertia of the complete disc is $I = \dfrac{1}{2}(4M)R^2 = 2MR^2$.

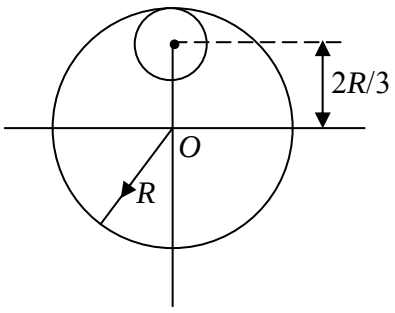
From symmetry each quarter of the disc has the same moment of inertia about the axis of rotation, hence required moment of inertia is $\dfrac{2MR^2}{4} = \dfrac{1}{2}MR^2$

Ans. 3: (a)

Solution: Total mass of the disc $= 9\ M$

Radius of disc $= R$

Moment of inertia of the complete disc about a perpendicular axis through O is



$I_1 = \dfrac{1}{2} \times 9M \times R^2 = \dfrac{9}{2}MR^2$

mass of the disc removed $= \dfrac{9M}{\pi R^2} \times \left(\dfrac{R}{3}\right)^2 = M$

by the parallel axes theorem, the moment of inertia of small disc about the axis through O,

$$I_2 = \frac{1}{2}M\left(\frac{R}{3}\right)^2 + M\left(\frac{2R}{3}\right)^2 = \frac{1}{2}MR^2$$

Moment of inertia of the remaining disc about a perpendicular axis through O,

$$I = I_1 - I_2 = \frac{9}{2}MR^2 - \frac{1}{2}MR^2 = 4MR^2$$

Ans. 4: (d)

Solution: The moment of inertia of the system about any side (say CD)

$I = M.I$ of A about $CD +$ moment of inertia of B about $CD + MI$ of C about $CD + MI$ of D about CD

$$= \left(\frac{2}{5}Ma^2 + Mb^2\right) + \left(\frac{2}{5}Ma^2 + Mb^2\right) + \frac{2}{5}Ma^2 + \frac{2}{5}Ma^2 = \frac{2}{5}M\left(4a^2 + 5b^2\right)$$

Ans. 5: (b)

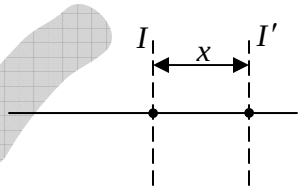
Solution: Let k_1 and k_2 be the radii of gyration of the ring and the disc respectively. Then

M.I of ring $= MR^2 = Mk_1^2$ or $k_1 = R$

M.I of disc $= \frac{1}{2}MR^2 = Mk_2^2$ or $k_2 = \frac{R}{\sqrt{2}}$ $\quad\therefore \frac{k_1}{k_2} = \sqrt{2} : 1$

Ans. 6: (d)

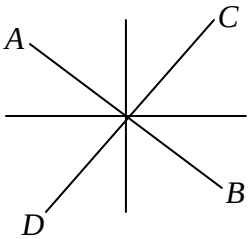
Solution: Using parallel axis theorem



$$I' = I + Mx^2 = \frac{ML^2}{12} + Mx^2$$

Ans. 7: (d)

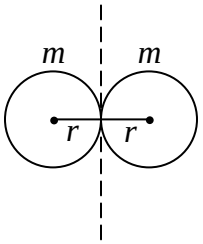
Solution: The M.I of the cross about a line perpendicular to the plane of the figure through the centre of cross is $\frac{Ml^2}{12} \times 2 = \frac{Ml^2}{6}$



The moment of inertia of the cross about the two bisectors are equal by symmetry and according to the theorem of perpendicular axes, the M.I of the cross about the bisector AB is $\frac{Ml^2}{12}$

Ans. 8: (c)

Solution: Using parallel axis theorem the M.I of the system about the tangent



$$= 2\left[\frac{2}{5}mr^2 + mr^2\right] = \frac{14mr^2}{5}$$

Ans. 9: (c)

Solution: Given that $I_Q = 4I_P$. The M.I. of a ring about its axis is $\text{mass} \times (\text{radius})^2$, let λ be the linear mass density of the wire, then

$$\lambda \times 2\pi(nr) \times n^2r^2 = 4\left(\lambda \times 2\pi r \times r^2\right) \Rightarrow n^2 = 4 \Rightarrow n = 4^{\frac{1}{3}}$$

Ans. 10: (c)

Solution: Let M be the mass of each disc. Let R_A and R_B be the radii of the discs A and B, respectively. Let t be the thickness of each disc. Then $M = \pi R_A^2 t d_A = \pi R_B^2 t d_B$ as $d_A > d_B$ so $R_A^2 < R_B^2$

Now $\frac{I_A}{I_B} = \frac{\frac{1}{2}MR_A^2}{\frac{1}{2}MR_B^2} = \frac{R_A^2}{R_B^2} < 1$ $\quad\therefore\quad I_A < I_B$

Ans. 11: (d)

Solution: Torque $\vec{\tau} = \vec{r} \times \vec{F}$. For central force $\vec{r}$ and $\vec{F}$ lies along the same line. Hence $\vec{\tau} = 0$. This implies that angular momentum is constant.

Ans. 12: (b)

Solution: Let M be the mass and r be the radius. Then moment of inertia $= \frac{2}{5}Mr^2$

Hence $L = \left(\frac{2}{5}Mr^2\right)\omega = \frac{2Mr^2\omega}{5}$

But since no net torque acts on the sphere angular momentum remains conserved. Hence increasing r, decreases ω. Increasing r also increases moment of inertia. Due to changing I and ω rotational kinetic energy also change

Ans. 13: (d)

Solution: Since no external torque acts on the system (Insect +disc), the angular momentum must remain constant. But since $L = I\omega$ and as the insect moves along the diameter I increases, hence ω must decrease, from centre to rim. From rim to centre I decreases, hence ω increases.

Ans. 14: (a)

Solution: The net force on the system is zero, hence there will be no translation. But since the net torque is non-zero, there will be rotation

Ans. 15 (b)

Solution: From the relation

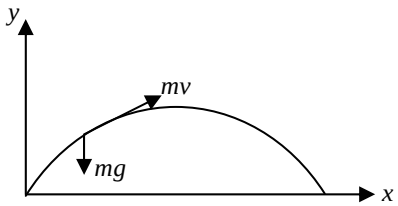
$\frac{d\vec{L}}{dt} = \vec{\tau}_{ext}$, when $\vec{L}$ is the total angular momentum and $\vec{\tau}$, the net external torque, we see that, when $\vec{\tau}_{ext} = 0$, $\vec{L}$ is constant

Ans. 16: (b)

Solution: The direction of angular velocity vector is always along the axis of rotation for a rigid body rotating about a fixed axis.

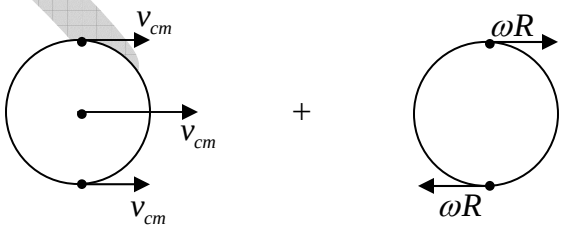
Ans. 17: (b)

Solution: About the initial position, the angular momentum of the particle is in the clockwise sense. Also the torque due to its weight mg is in the clockwise sense. Hence the angular momentum increases.



Ans. 18 (a)

Solution: The net velocity is the sum of velocity due to translation and velocity due to rotation



For rolling we also have $v_{cm} = \omega R$. Hence $v_{top} = v_{cm} + v_{cm} = 2v_{cm}$ and $v_{bottom} = v_{cm} - v_{cm} = 0$

Ans. 19: (a)

Solution: The acceleration of a rolling body down an incline is given b

$$a = \frac{g\sin\theta}{1+\frac{I}{MR^2}}$$

For hollow cylinder $\frac{I}{MR^2} = 1$

For solid cylinder $\frac{I}{MR^2} = \frac{1}{2}$. Hence $a_{\text{hollow}} < a_{\text{solid}}$

Since acceleration of solid cylinder is greater if will reach earlier

Ans. 20 (d)

Solution: The torque acting on a particle is $\vec{\tau} = \vec{r} \times F$ and not $\vec{F} \times \vec{r}$

The relation $\vec{\tau} = \frac{d\vec{L}}{dt}$ says zero net torque means constant angular momentum. It also says changing angular momentum necessarily gives rise to torque. Neither zero net force implies zero net torque nor the zero net torque implies zero net force

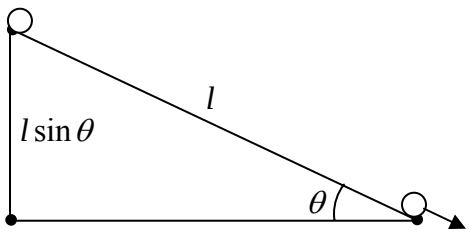
Ans.21: (a)

Solution: From the conservation of mechanical energy

$$Mgl\sin\theta = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 \Rightarrow Mgl\sin\theta = \frac{1}{2}Mv^2 + \frac{1}{2}.\frac{MR^2}{2}.\frac{v^2}{R^2} \quad [\text{For rolling } v = \omega R]$$

$$gl\sin\theta = \frac{v^2}{2} + \frac{v^2}{4}$$

$$\Rightarrow \frac{3v^2}{4} = gl\sin\theta \Rightarrow v = \sqrt{\frac{4gl\sin\theta}{3}}$$



Ans.22: (d)

Solution: Due to forward slipping $(v_0 > \omega_0 r)$ the frictional force is backward

The translational of velocity at time t

$$v(t) = v_0 - \frac{f}{M}t \qquad \text{(i)}$$

The angular velocity at time t

$$\omega(t)=\frac{v_0}{2r}+\alpha t=\frac{v_0}{2r}+\frac{fr}{\frac{2Mr^2}{5}}t \Rightarrow \omega(t)=\frac{v_0}{2r}+\frac{5f}{2Mr}t$$

Rolling will start when $v(t)=r\omega(t)$

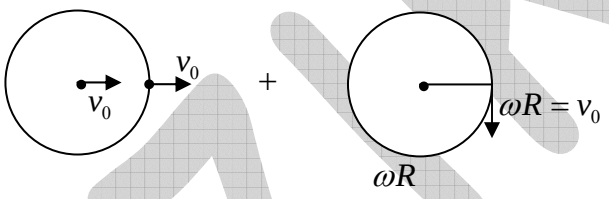
$$\Rightarrow v(t)=\frac{v_0}{2}+\frac{5f}{2M}t \qquad \text{(ii)}$$

Eliminating t from (i) and (ii)

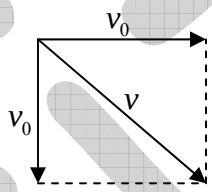
$$\frac{5}{2}v(t)+v(t)=\frac{5}{2}v_0+\frac{v_0}{2} \text{ or } v(t)=\frac{6v_0}{7}$$

Ans.23: (c)

Solution: Net velocity of a particle = translational velocity + rotational velocity



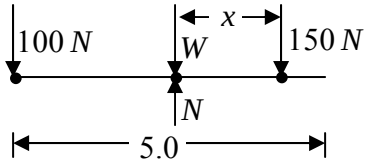
Hence



$$\therefore v=\sqrt{v_0^2+v^2}=\sqrt{2}v_0$$

Ans.24: (d)

Solution: The forces acting on the seesaw are shown in the figure.



Taking torques about the fulcrum

$150\times x=100\times 2.5$ which gives $x=1.7\ m$

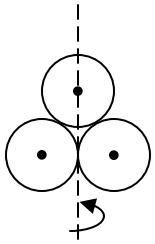
Ans. 25: (d)

Solution: The velocity of the point of contact of rolling body is zero. Since v and ω are constants, the only acceleration of the bottommost point is centripetal acceleration $\frac{v^2}{R}$

NAT (Numerical Answer type)

Ans. 26: 140

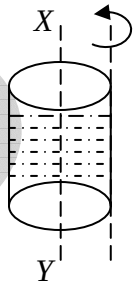
Solution: The M.I. of the system about the axis = sum of moment of inertia of individual rings.



$$I = \frac{3}{2}mr^2 + \frac{3}{2}mr^2 + \frac{mr^2}{2} = \frac{7}{2}mr^2 = \frac{7}{2}\times 10\times 4 = 140\, kg\cdot m^2 .$$

Ans. 27: 135

Solution: The moment of inertia of the cylinder about its axis $= \frac{MR^2}{2}$



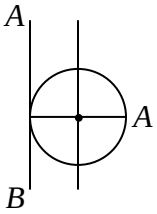
Using parallel axis theorem

$$I = I_{XY} + MR^2 = \frac{MR^2}{2} + MR^2 = \frac{3}{2}MR^2$$

$$= \frac{3}{2}\times 10\times 9 = 135\, kg\cdot m^2$$

Ans. 28: 2000

Solution: The moment of inertia of the disc about axis AB is



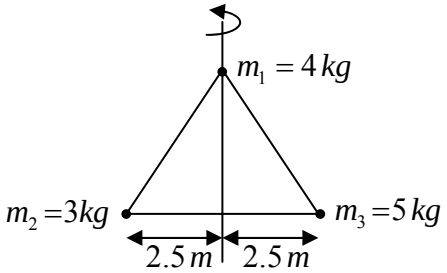
$$I_{AB} = I_{A'B'} + MR^2 = \frac{MR^2}{4} + MR^2 = \frac{5}{4}MR^2$$

The angular speed ω of the disc $= \frac{v}{R}$

Hence $K = \frac{1}{2}\times\frac{5}{4}MR^2\times\frac{v^2}{R^2} = \frac{1}{2}\times\frac{5}{4}\times 8\times 400 = 2000\ J$

Ans. 29: 50

Solution: The moment of inertia of the system about the altitude passing through m_1 is



$$I = m_1r_1^2 + m_2r_2^2 + m_3r_3^2$$

$$= (4\,kg)(0)^2 + (3\,kg)(2.5\,m)^2 + (5\,kg)(2.5\,m)^2$$

$$= (2.5)^2\times 8 = 50\, kg\cdot m^2$$

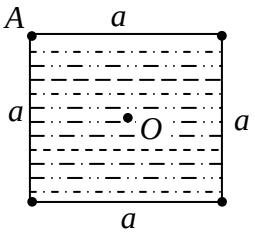
Head office
fiziks, H.No. 23, G.F, Jia Sarai,
Near IIT, Hauz Khas, New Delhi-16
Phone: 011-26865455/+91-9871145498

Branch office
Anand Institute of Mathematics,
28-B/6, Jia Sarai, Near IIT
Hauz Khas, New Delhi-16

Website: www.physicsbyfiziks.com
Email: fiziks.physics@gmail.com

Ans. 30: 64

Solution: The moment of inertia of the plate about an axis perpendicular to the plate and passing through centre O is



$$= \frac{m}{12}\left(a^2 + a^2\right) = \frac{ma^2}{6}$$

By parallel axis theorem, the moment of inertia of the plate about an axis through one of its corners,

$$I_A = I_0 + m\left(\frac{a}{\sqrt{2}}\right)^2 = \frac{ma^2}{6} + \frac{ma^2}{2} = \frac{2}{3}ma^2$$

$$= \frac{2}{3} \times 6 \times 16 = 64\, kg \cdot m^2$$

Ans. 31: $2\sqrt{26}$

Solution: The angular momentum of the disc

$\vec{L} = \vec{L}_{cm} + M\left(\vec{R} \times \vec{V}\right)$ when $\vec{L}_{cm}$ is the angular momentum of the disc with respect to centre of mass, $\vec{R}$ the position of centre of mass and $\vec{V}$ the velocity of centre of mass

$\vec{L}_{cm} = I_{cm}\vec{\omega}$

or $\vec{L}_{cm} = \frac{1}{2}.(2)(1)^2\left(-10\hat{k}\right) = -10\hat{k}$

$M\left(\vec{R} \times \vec{V}\right) = 2\left[\left(X\hat{i} + Y\hat{j}\right) \times (10 \times 1)\hat{i}\right] = -2Y\hat{k} = -2 \times 1\hat{k} = -2\hat{k}$

Hence $L = \sqrt{(-10)^2 + (-2)^2} = \sqrt{104} = 2\sqrt{26}$ units

Ans. 32: 5.98

Solution: The acceleration of a rolling body down an incline is given by the formula

$$a = \frac{g \sin\theta}{1 + \frac{I}{MR^2}}$$

$$\therefore a = \frac{10 \times \frac{3}{5}}{1 + \frac{2}{5 \times 100}} = \frac{6}{\frac{251}{250}} = 5.98 \ m/s^2$$

Ans. 33: 100

Solution: From the conservation of mechanical energy

$$\frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 = 0 + 0 + Mgh$$

or $$\frac{1}{2}Mv^2 + \frac{1}{2}.\frac{2}{5}MR^2.\frac{v^2}{R^2} = Mgh$$

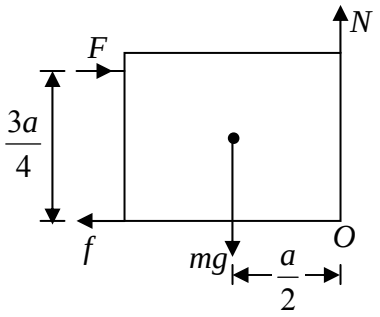
$$\frac{v^2}{2} + \frac{v^2}{5} = gh \Rightarrow \frac{7v^2}{10} = gh$$

$\therefore$ $$v = \sqrt{\frac{10gh}{7}}$$

or $$v = \sqrt{\frac{10 \times 10 \times 700}{7}} = 100 \ m/s$$

Ans. 34: 200 N

Solution: Moments of normal reaction and frictional force about the point of tipping O is zero. At the time of tipping N passes through O. To tip about the edge, moment of F should be greater than the moment of mg

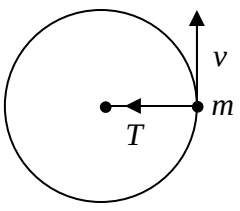


or $$F\left(\frac{3a}{4}\right) > mg\left(\frac{a}{2}\right) \text{ or } F > \frac{2}{3}mg$$

Hence $F > \frac{2}{3}.30 \times 10 \Rightarrow F > 200\,N$

Ans. 35: –3

Solution: $mvr = k$ (a constant) $\Rightarrow v = \frac{k}{mr}$



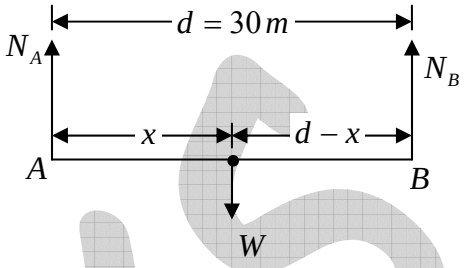
$$T = \frac{mv^2}{r} = \frac{m}{r}\left(\frac{k}{mr}\right)^2 = \frac{k^2}{m}.\frac{1}{r^3}$$

$$T = Ar^{-3} \qquad (\text{where } A = \frac{k^2}{m})$$

Hence $n = -3$

Ans. 36: $\frac{40}{3}$

Solution: Taking torque about point B



$$W(d-x) = N_A.d$$

$$\Rightarrow N_A = \left(\frac{d-x}{d}\right)w = \frac{30-10}{30}\times 20 \Rightarrow N_A = \frac{40}{3}\ N$$

Ans. 37: 8

Solution: Since no net external torque acts on the system, hence, the angular momentum about the axis of rotation remains conserved. Initially the two discs rotate with different angular speeds but due to friction they acquire common angular speed. From $L = I\omega$ we have $I_i\omega_i = I_f\omega_f$

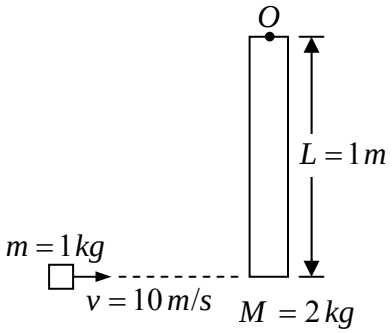
$$\Rightarrow \qquad \frac{1}{2}MR^2(10) = \left[\frac{1}{2}MR^2 + \frac{1}{2}.\frac{M}{4}.R^2\right]\omega_f$$

$$5MR^2 = \frac{5MR^2}{8}\omega_f$$

or $\omega_f = 8\,rad/s$

Ans. 38: 6

Solution: Angular momentum of the system about an axis perpendicular to the plane of paper and passing through O will remains conserved.



$$L_i = L_f$$

$$\therefore mvL = I\omega = \left(mL^2 + \frac{ML^2}{3}\right)\omega$$

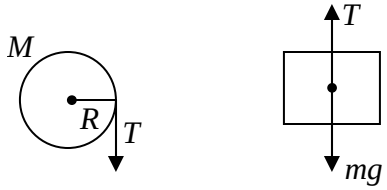
$$\therefore \omega = \frac{3mv}{L(3m+M)} = \frac{3\times 1\times 10}{1(3+2)}$$

$$\omega = 6\,rad/s^2$$

Ans. 39: 4

Solution: For the block $mg - T = ma$ (i)

For the pulley



$TR = I\alpha$, α being the angular acceleration

$$\Rightarrow TR = \frac{MR^2}{2}.\frac{a}{R} \Rightarrow T = \frac{Ma}{2} \quad \text{(ii)}$$

From (i) and (ii) $2g - \frac{Ma}{2} = ma$

$$\Rightarrow a\left[m + \frac{M}{2}\right] = mg \Rightarrow a\left(\frac{2m+M}{2}\right) = mg$$

$$a = \frac{2mg}{2m+M} = \frac{2\times 2\times 10}{4+6} \quad a = 4\ m/s^2$$

Ans. 40: 1

Solution: The tension in the rope is the only net force on the block and it does not exert any torque about the axis of rotation. Hence, the angular momentum of the block about the axis should remains conserved.

Hence $m_1 v_1 r_1 = m_2 v_2 r_2$ (i)

When r_2 and v_2 are the radius and speed when the string breaks. Let T_2 be the tension when string breaks.

Hence $T_2 = \frac{mv_2^2}{r_2}$ (ii)

From (i) and (ii) $r_2 = \left(\frac{mv_1^2 r_1^2}{T_2}\right)^{\frac{1}{3}}$

$$r_2 = \left(\frac{150\times 16\times 0.25}{600}\right)^{\frac{1}{3}} \Rightarrow r_2 = 1\ m$$

MSQ (Multiple Select Questions)

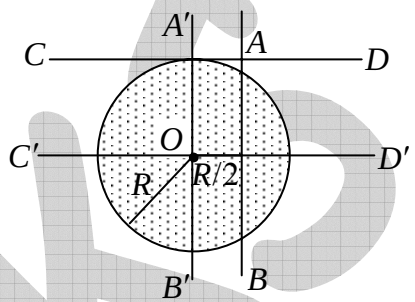
Ans. 41: (a), (b) and (c)

Solution: From symmetry consideration and perpendicular axes theorem

$$I_1 = I_2 = I_3 = I_4 = \frac{ML^2}{12}$$

Ans. 42: (a), (b) and (c)

Solution: The moment of inertia of the disc about point O is $\frac{MR^2}{2}$.



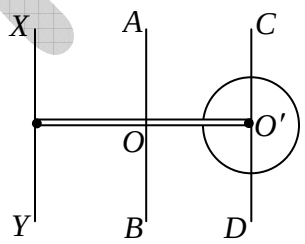
Moment of inertia about AB is

$$I_{AB} = I_{A'B'} + M\left(\frac{R}{2}\right)^2 = \frac{MR^2}{4} + \frac{MR^2}{4} = \frac{MR^2}{2}$$

$$I_{CD} = I_{C'D'} + MR^2 = \frac{MR^2}{4} + MR^2 = \frac{5}{4}MR^2$$

Ans. 43: (a), (b) and (c)

Solution: From parallel axis theorem, M.I. of the rod about XY is



$$I_{rod} = I_{AB} + m\left(\frac{l}{2}\right)^2 = \frac{ml^2}{12} + \frac{ml^2}{4}$$

$$\Rightarrow \quad I_{rod} = \frac{ml^2}{3}$$

The moment of inertia of the sphere (using parallel axis theorem) about $X\,Y$ is

$$I_{sphere} = I_{CD} + ml^2$$

$$\Rightarrow \quad I_{sphere} = \frac{2}{5}mr^2 + ml^2$$

The moment of inertia of the system about $X\,Y$ is

$$I_{system} = \frac{ml^2}{3} + \frac{2}{5}mr^2 + ml^2 = \frac{ml^2}{3} + \left(\frac{2}{5}mr^2 + ml^2\right)$$

Ans. 44: (c) and (d)

Solution: Moment of inertia of the system about AB = Moment of inertia of the hollow sphere about AB + Moment of inertia of solid sphere about AB

$$I_{AB} = \left[\frac{2}{3}MR^2 + M\left(\frac{d}{2}\right)^2\right] + \left[\frac{2}{5}MR^2 + M\left(\frac{d}{2}\right)^2\right]$$

$$= \frac{16}{15}MR^2 + \frac{Md^2}{2}$$

Moment of inertia of the system about CD = Moment of inertia of the hollow sphere about CD + Moment of inertia of the solid sphere about CD.

$$\Rightarrow \qquad I_{CD} = \left[\frac{2}{3}MR^2 + MR^2\right] + \left[\frac{2}{5}MR^2 + MR^2\right] = \frac{16}{15}MR^2 + 2MR^2$$

Ans. 45: (a), (b), (c) and (d)

Solution: For the disc about its diameter $Mk^2 = \frac{MR^2}{4} \Rightarrow k = \frac{R}{2}$

For the disc about the tangent in its plane $Mk^2 = \frac{MR^2}{4} + MR^2 = \frac{5}{4}MR^2 \Rightarrow k = \frac{\sqrt{5}}{2}R$

For the thin rod about an axis through one end $MK^2 = \frac{ML^2}{3} \Rightarrow K = \frac{L}{\sqrt{3}}$

For a rectangular lamina about an axis through the centre and perpendicular to the rod

$Mk^2 = \frac{M}{12}\left(l^2 + b^2\right) \Rightarrow k = \frac{1}{2}\sqrt{\frac{l^2 + b^2}{3}}$.

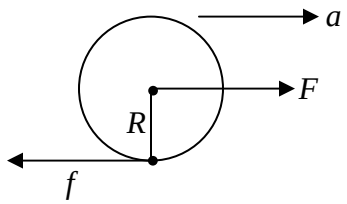
Ans. 46: (a), (b), (c) and (d)

Solution: The force F tries to pull the point of contact in the forward direction, hence the frictional force is backward.

Translation equation

$F - f = Ma$ (i)

Rotational equation about the centre of mass



$$fR = I\alpha = \frac{2MR^2}{5}.\frac{a}{R}$$

$$\Rightarrow f = \frac{2Ma}{5} \qquad \text{(ii)}$$

From (i) and (ii) $F = \frac{7Ma}{5}$ or $a = \frac{5F}{7M}$

and $f = \frac{2M}{5}.\frac{5F}{7M} = \frac{2F}{7}$

Ans. 47: (c) and (d)

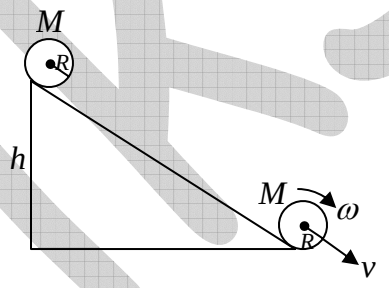
Solution: Consider an arbitrary sphere of mass M and radius R rolling down an incline from height h.

From conservation of mechanical energy

$$Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$$

$$\Rightarrow \quad Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}.\frac{2MR^2}{5}.\frac{v^2}{R^2}$$

$$\Rightarrow \quad gh = \frac{7v^2}{10} \quad \text{or } v = \sqrt{\frac{10gh}{7}}$$



This is independent of mass and radius thus all the spheres reach with the same speed.

The total kinetic energy of the spheres of the bottom is equal to the loss in initial potential energy. Hence

$K = Mgh \Rightarrow K \propto M$

since sphere (ii) has the largest mass if will have the largest kinetic energy

Ans. 48: (c) and (d)

Solution: All points in the body, in plane perpendicular to axis of rotation, revolve in concentric circles.

All points lying on the circle of same radius have same speed (and also same magnitude of acceleration) but different directions of velocity.

Angular speed of body of any instant with respect to any point in the body is the same by definition.

Ans. 49: (a) and (b)

Solution: In case of rolling the point of contact and the surface on which the object is rolling should have save velocity.

Hence $v_0 - r\omega_0 = 3v$ (i)

and $v_0 + r\omega_0 = -v$ (ii)

From (i) and (ii) $v_0 = v$ and $\omega_0 = \dfrac{2v}{r}$

Ans. 50: (a) and (b)

Solution: $v(t) = \dfrac{f}{M}t$ and $\omega(t) = \omega_0 - \dfrac{fR}{\frac{MR^2}{2}}t$

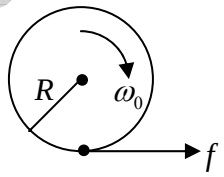
$$v(t) = \frac{\mu_0 Mg}{M}t = \mu_0 gt \text{ or } \omega(t) = \omega_0 - \frac{2\mu_0 MgR}{MR^2}t = \omega_0 - \frac{2\mu_0 g}{R}t$$

when rolling being

$$v(t) = R\omega(t) \Rightarrow \mu_0 gt = \omega_0 R - 2\mu_0 gt \Rightarrow t = \frac{\omega_0 R}{3\mu_0 g}$$

velocity at the time of rolling

$$v(t) = \mu_0 g.\frac{\omega_0 R}{3\mu_0 g} = \frac{\omega_0 R}{3}$$



displacement $= \dfrac{1}{2}.\mu_0 g\left(\dfrac{\omega_0 R}{3\mu_0 g}\right)^2 = \dfrac{\omega_0^2 R^2}{18\mu_0 g}$

work done by force of friction $= (-\mu_0 Mg) \times \dfrac{\omega_0^2 R^2}{18\mu_0 g} = -\dfrac{\omega_0^2 R^2 M}{18}$

Ans. 51: (a), (c) and (d)

Solution: Since initial linear momentum of the system is zero, and no external force acts on the system,

hence from $\vec{P} = M\vec{v}_{cm}, v_c = 0$

Since no external torque acts on the system $L_i = L_f$

$\Rightarrow (2mv)a + (2mv)2a = I\omega$

$\Rightarrow 6mva = I\omega$ (i)

$$I = \frac{(8m)(6a)^2}{12} + m(2a)^2 + 2m(a^2) = 30ma^2$$

hence from equation (i) $\omega = \frac{v}{5a}$

Further, $E = \frac{1}{2}I\omega^2 = \frac{1}{2}.30ma^2.\frac{v^2}{25a^2} = \frac{3mv^2}{5}$

Ans. 52: (a), (b) and (c)

Solution: $\vec{\tau} = \vec{A}\times\vec{L}$ i.e. $\frac{d\vec{L}}{dt} = \vec{A}\times\vec{L}$

This implies that $\frac{d\vec{L}}{dt}$ is perpendicular to both $\vec{A}\times\vec{L}$ of all instants.

We have $\vec{L}.\vec{L} = L^2$ $\Rightarrow \vec{L}.\frac{d\vec{L}}{dt} + \frac{d\vec{L}}{dt}.\vec{L} = 2L\frac{dL}{dt}$

then $\frac{dL}{dt} = 0$

or magnitude of $\vec{L}$ does not charge with time. Similarly we can show that $\vec{L} \perp \vec{A}$. Thus the component of $\vec{L}$ along $\vec{A}$ is always constant.

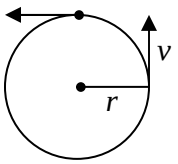
Ans. 53: (b) and (c)

Solution: $\vec{v}_A = 0,\ \vec{v}_B = v\hat{i},\ \vec{v}_C = 2v\hat{i}$

Ans. 54: (d)

Solution: The magnitude of angular momentum of any instant is $L = mvr$

Since r is the same at all time and v is decreasing, L is decreasing. The acceleration $\vec{a}$ is towards the centre only when the particle moves in the circle with constant speed.

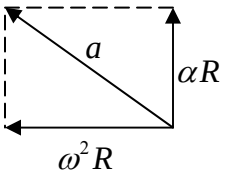


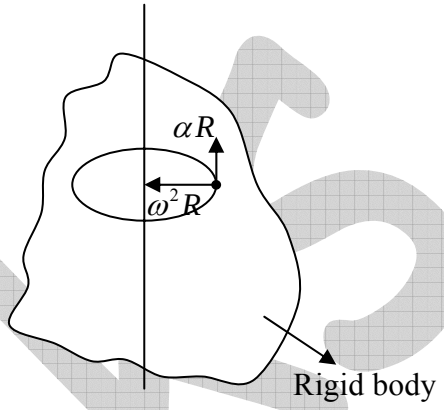
From the figure, we see that $\vec{r}\times\vec{P}$ is always out of paper, hence direction of $\vec{L}$ remains constant.

Ans. 55: (a) and (b)

Solution: For a fixed axis rotation, the net torque is always along the axis of rotation, otherwise the axis itself would not remain fixed.

All the particles of the rigid body moves equal angles in equal time. Hence angular velocity of all particles in the same





The linear acceleration a depends on the distance of the particle from the axis of rotation.

The angular momentum vector lies in the direction of angular velocity only if the axis rotation is one of the symmetry axes of the body.

5. Fluid Mechanics

Fluids

Matter is broadly divided into three categories, solid, liquid and gas. The intermolecular forces are strong in solids, so that the shape and size of solids do not easily change. This force is comparatively less in liquids and so the shape is easily changed. Although the shape of a liquid can be easily changed, the volume of a given mass of a liquid is not so easy to change. It needs quite a good effort to change the density of liquids. In gases, the intermolecular forces are very small and it is simple to change both the shape and the density of a gas.

We shall assume that the liquids we deal with are incompressible and non viscous. The first condition means that the density of the liquid is independent of the variations in pressure and always remains constant. The second condition means that parts of the liquid in contact do not exert any tangential force on each other. The force by one part of the liquid on the other part is perpendicular to the surface of contact. Thus, there is no friction between the adjacent layers of a liquid.

5.1 Pressure in a Fluid

Consider a point A in the fluid (figure 1). Imagine a small area ΔS containing the point A. The fluid on one side of the area presses the fluid on the other side and vice versa. Let the common magnitude of the forces be F. Then the pressure of the fluid at the point A as

$$P = \lim_{\Delta S \to 0} \frac{F}{\Delta S} \qquad \ldots(1)$$

For a homogeneous and nonviscous fluid, this quantity does not depend on the orientation of ΔS and hence we talk of pressure at a point. For such a fluid, pressure is a scalar quantity having only magnitude.

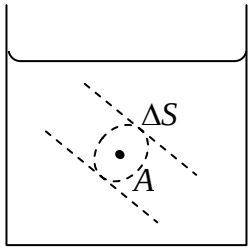


Figure 1

Unit of Pressure

The SI unit of pressure is N/m^2 called Pascal (Pa).

5.1.1 Variation of Pressure with Height

Let us consider two points A and B (figure 2) separated by a small vertical height dz. Imagine a horizontal area ΔS_1 containing A and an identical horizontal area ΔS_2 containing B. The area $\Delta S_1 = \Delta S_2 = \Delta S$. Consider the fluid enclosed between the two surfaces $\Delta S_1, \Delta S_2$ and the vertical boundary joining them. The vertical forces acting on this fluid are

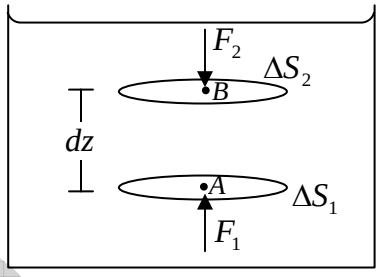


Figure 2

(a) F_1, vertically upward by the fluid below it

(b) F_2, vertically downward by the fluid above it and

(c) weight W, vertically downward

Let the pressure at the surface A be P and the pressure at B be $P + dP$. Then

$$F_1 = P\Delta S \quad \text{and} \quad F_2 (P + dP)\Delta S .$$

The volume of the fluid considered is $(\Delta S)(dz)$. If the density of the fluid at A is ρ, the mass of the fluid considered is $\rho(\Delta S)(dz)$ and hence its weight W is $W = \rho(\Delta S)(dz)g$.

For vertical equilibrium,

$$F_1 = F_2 + W$$

or $$P\Delta S = (P + dP)\Delta S + \rho g(dz)\Delta S$$

or $$dP = -\rho g(dz). \qquad \ldots(2)$$

As we move up through a height dz the pressure decreases by $\rho g\, dz$ where ρ is the density of the fluid at that point.

Now consider two points at $z = 0$ and $z = h$. If the pressure at $z = 0$ is P_1 and that at $z = h$ is P_2, then from equation (2),

$$\int_{P_1}^{P_2} dP = \int_0^z -\rho g\, dz \;\Rightarrow P_2 - P_1 = -\rho g z$$

$$\Rightarrow P_1 = P_2 + \rho g z \qquad \ldots.(3)$$

Next consider two points A and B in the same horizontal line inside a fluid. Imagine a small vertical area ΔS_1 containing the point A and a similar vertical area ΔS_2 containing the point B.

Figure 3

The area $\Delta S_1 = \Delta S_2 = \Delta S$. Consider the contained in the horizontal cylinder bounded and ΔS_1 and ΔS_2. If the pressures at A and B are P_1 and P_2 respectively, the forces in the direction AB are

(a) $P_1 \Delta S$ towards right and (b) $P_2 \Delta S$ towards left.

If the fluid remains in equilibrium, $P_1 \Delta S = P_2 \Delta S$ or $P_1 = P_2$

Thus, the pressure is same at two in the same horizontal level.

5.1.2 Pascal's Law

We have seen in the previous section that the pressure difference between two points in a liquid at rest depends only on the difference in vertical height between the points. The difference is in fact $\rho g z$, where ρ is the density of the liquid (assumed constant) and z is the difference in vertical height.

If the pressure in a liquid is changed at a particular point, the change is transmitted to the entire liquid without being diminished in magnitude.

As an example, suppose a flask fitted with a piston is filled with a liquid as shown in figure (4). Let an external force F be applied on the piston. If the cross-sectional area of the piston is A, the pressure just below the piston is increased by F/A. By Pascal's law, the pressure at any point B will also increase by the same amount F/A. This is because the pressure at B has to be $\rho g z$ more than the pressure at the piston, where z is the vertical distance of B below the piston.

By applying the force we do not appreciably change z (as the liquid is supposed to be incompressible) and hence the pressure difference remains unchanged. As the pressure at the piston is increased by F/A, the pressure at B also increases by the same amount.

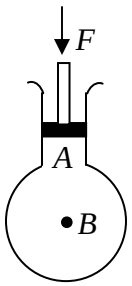


Figure 4

5.1.3 Atmospheric Pressure and Barometer

The atmosphere of the earth is spread upto a height of about 200 km. This atmosphere presses the bodies on the surface of the earth. The force exerted by the air on any body is perpendicular to the surface of the body. Consider a small surface ΔS in contact with air. If the force exerted by the air on this part is F, the atmospheric pressure is $P_0 = \lim_{\Delta S \to 0} \frac{F}{\Delta S}$

Atmospheric pressure at the top of the atmosphere is zero as there is nothing above it to exert the force. The pressure at a distance z below the top will be $\int_0^z \rho g \, dz$. Remember, neither ρ nor g can be treated as constant over large variations in heights. However, the density of air is quite small and so the atmospheric pressure does not vary appreciably over small distances. Thus, we say that the atmospheric pressure at Patna is 76 cm of mercury without specifying whether it is at Gandhi Maidan or at the top of Golghar.

Torricelli devised an ingenious way to measure the atmospheric pressure. The instrument is known as barometer. In this, a glass tube open at one end and having a length of about a meter is filled with mercury. The open end is temporarily closed (by a thumb or otherwise) and the tube is inverted in a cup of mercury. With the open end dipped into the cup, the temporary closure is removed. The mercury column in the tube falls down a little and finally stays there.

Figure (5) shows schematically the situation. The upper part of the tube contains vacuum as the mercury goes down and no air is allowed in.

Thus, the pressure at the upper end A of the mercury column inside the tube is $P_A = 0$. Let us consider a point C on the mercury surface in the cup and another point B in the tube at the same horizontal level. The pressure at C is equal to the

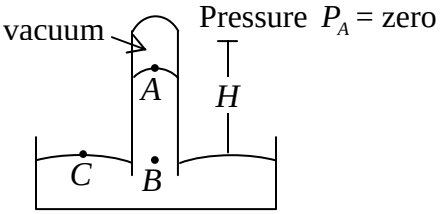


Figure 5

atmospheric pressure. As B and C are in the same horizontal level, the pressures at B and C are equal. Thus, the pressure at B is equal to the atmospheric pressure P_0 in the lab.

Suppose the point B is at a depth H below A. If ρ be the density of mercury,

$$P_B = P_A + \rho g H \quad \text{or,} \quad P_0 = \rho g H. \qquad \ldots (4)$$

The height H of the mercury column in the tube above the surface in the cup is measured. Knowing the density of mercury and the acceleration due to gravity, the atmospheric pressure can be calculated using equation (4).

The atmospheric pressure is often given as the length of mercury column in a barometer. Thus, a pressure of 76 cm of mercury means

$$P_0 = \left(13.6\times10^3\, kg/m^3\right)\left(9.8 m/s^2\right)\left(0.76 m\right)$$

$$= 1.01\times10^5\, Pa.$$

This pressure is written as 1 atm. If the tube is insufficient in length, the mercury column will not fall down and no vacuum will be created. The inner surface of the tube will be in contact with the mercury at the top and will exert a pressure P_A on it.

Example: Water is filled in a flask upto a height of 20 cm. The bottom of the flask is circular with radius 10 cm. If the atmospheric pressure is $1.01\times10^6\, Pa$, find the force exerted by the water on the bottom. Take $g = 10 m/s$ and density of water $= 1000\, kg/m^3$.

Solution: The at the surface of the water is equal to the atmospheric pressure P_0. The pressure at the bottom is

$$P = P_0 + h\rho g \quad = 1.01\times10^6\, Pa + \left(0.20\, m\right)\left(1000\, kg/m^3\right)\left(10 m/s^2\right)$$

$$= 1.01\times10^6\, Pa + 0.02\times10^5\, Pa = 1.03\times10^6\, Pa\,.$$

The area of the bottom

$$= \pi r^2 = 3.14\times\left(0.1m\right)^2 = 0.0314 m^2$$

The force on the bottom is, therefore, $F = P\pi r^2$

$$= \left(1.03\times10^5\, Pa\right)\times\left(0.0314\, m^2\right) = 3230 N$$

5.1.4 Manometer

Manometer is a simple device to measure the pressure in a closed vessel containing a gas. It consists of a $U-$tube having some liquid. One end of the tube is open to the atmosphere and the other end is connected to the vessel (figure 6).

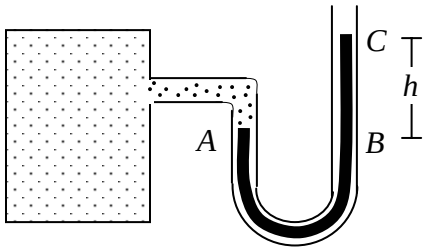


Figure 6

The pressure of the gas is equal to the pressure at A

= pressure at B

= pressure at $C + h\rho g = P_0 + h\rho g$

where P_0 is the atmospheric pressure, $h = BC$ is the difference in levels of the liquid in the two arms and ρ is the density of the liquid.

The excess pressure $P - P_0$ is called the gauge pressure.

5.2 Archimedes' Principle

When a body is partially or fully dipped into a fluid, the fluid exerts forces on the body. At any small portion of the surface of the body, the force by the fluid is perpendicular to the surface and is to the pressure at that point multiplied by the area (figure 8). The resultant of all these contact forces is called the force of buoyancy or buoyant force.

Archimedes' principle states that when a body is partially or fully dipped into a fluid at rest, the fluid exerts an upward force of buoyancy equal to the weight of the displaced fluid.

Archimedes' principle is not an independent principle and may be deduced from Newton's laws of motion.

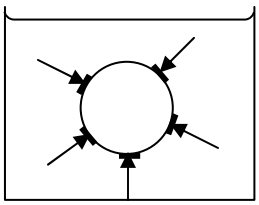


Figure 7

Consider the situation shown in figure (7) where a body is shown dipped into a fluid. Suppose the body dipped in the fluid is replaced by the same 'fluid of equal volume. As the entire fluid now becomes homogeneous, all parts will remain in equilibrium. The part of the fluid substituting the body also remains in equilibrium.

Forces acting on this substituting fluid are

(a) the weight mg of this part of the fluid and

(b) the resultant B of the' contact forces by the remaining fluid.

As the substituting fluid is in equilibrium, these two should be equal and opposite. Thus,

$$B = mg \qquad \ldots (5)$$

and it acts in the vertically upward direction.

Floatation

When a solid body is dipped into a fluid, the fluid exerts an upward force of buoyancy on the solid. If the force of buoyancy equals the weight of the solid, the solid will remain in equilibrium. This is called floatation. When the overall density of the solid is smaller than the density of the fluid, the solid floats with a part of it in the fluid. The fraction dipped is such, that the weight of the displaced fluid equals the weight of the solid.

Example: A $700\,g$ solid cube having an edge of length $10\,cm$ floats in water. How much volume of the cube is outside the water? Density of water $= 1000\,kg/m^3$.

Solution: The weight of the cube is balanced by the buoyant force. The buoyant force is equal to the weight of the water displaced. If a volume V of the cube is inside the water, the weight of the displaced water $= v\rho g$, where p is the density of water. Thus,

$$V\rho g = (0.7\,kg)g$$

or
$$V = \frac{0.7\,kg}{\rho} = \frac{0.7\,kg}{1000kg/m^3} = 7\times10^{-4}\,m^3 = 700\,cm^3$$

The total volume o the cube $= (10\,cm)^3 = 1000\,cm^3$

The volume outside the water is

$$100\,cm^3 - 700\,cm^3 = 300\,cm^3.$$

5.2.1 Pressure Difference and Buoyant Force in Accelerating Fluids

Equations (3) and (5) were derived by assuming that the fluid under consideration is in equilibrium in an inertial frame. If this is not the case, the equations must be modified. We shall discuss some special cases of accelerating fluids.

5.2.2 A Liquid Placed in an Elevator

(a) Pressure Difference

Suppose a beaker contains some liquid and it is placed in an elevator which is going up with acceleration a_0 (figure 8). Let A and B be two points in the liquid, B being at a vertical height z above A. Construct a small horizontal area ΔS around A and an equal horizontal area around B. Construct a vertical cylinder with the two areas as the faces. Consider the motion of the liquid contained within this cylinder. Let P_1 be the pressure at A and P_2 be the pressure at B.

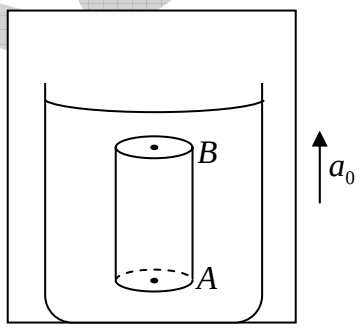


Figure 8

Forces acting on the liquid contained in the cylinder, in the vertical direction, are

(a) $P_1\Delta S$, upward due to the liquid below it

(b) $P_2\Delta S$, downward due to the liquid above it and

(c) weight $mg = (\Delta S)z\rho g$ downward, where ρ is the density of the liquid.

Under the action of these three forces the liquid is accelerating upward with an acceleration a_0. From Newton's second law

$$P_1\Delta S - P_2\Delta S - mg = ma_0$$

or $$(P_1 - P_2)\Delta S = m(g + a_0) = (\Delta S)z\rho(g + a_0)$$

or $$P_1 - P_2 = \rho(g + a_0)z. \qquad \ldots(6)$$

(b) Buoyant Force

Now suppose a body is dipped inside a liquid of density ρ placed in an elevator going up with acceleration a_0. Let us calculate the force of buoyancy B on this body. As was done earlier, let us suppose that we substitute the body into the liquid by the same liquid of equal volume. The entire liquid becomes a homogenous mass and hence the substituted liquid is at rest with respect to the rest of the liquid. Thus, the substituted liquid is also going up with an acceleration a together with the rest of the liquid.

The forces acting on the substituted liquid are

(a) the buoyant force B and

(b) the weight mg of the substituted liquid. From Newton's second law,

From Newton's second law,

$$B - mg = ma_0 \quad \text{or} \quad B = m(g + a_0) \qquad \ldots.(7)$$

5.2.3 Free Surface of a Liquid in Horizontal Acceleration

Consider a liquid placed in a beaker which is accelerating horizontally with acceleration a_0 (figure 9). Let A and B be two points in the liquid at a separation l in the same horizontal line along the acceleration a_0. We shall first obtain the pressure difference between the points A and B

Construct a small vertical area ΔS around A and an equal area around B. Consider the liquid contained in the horizontal cylinder with the two areas as the flat faces. Let the pressure at A be P_1 and the pressure at B be P_2. The forces along the line AB are

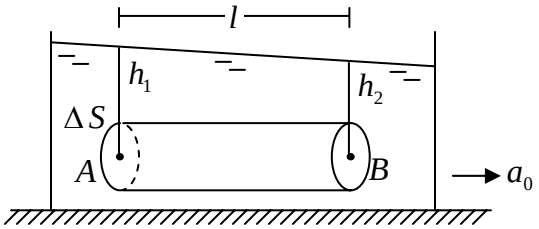


Figure 9

(a) $P_1 \Delta S$ towards right due to the liquid on the left and

(b) $P_2 \Delta S$ towards left due to the liquid on the right.

Under the action of these forces, the liquid contained in the cylinder is accelerating towards right. From Newton's second law,

$$P_1\Delta S - P_2\Delta S = ma_0 \quad \text{or} \quad (P_1 - P_2)\Delta S = (\Delta S)l\rho a_0$$

or $$P_1 - P_2 = l\,\rho\, a_0 \qquad \ldots. (8)$$

The two points in the same horizontal line do not have equal pressure if the liquid is accelerated horizontally.

As there is no vertical acceleration, the equation (3) is valid. If the atmospheric pressure is P_0, the pressure at A is $P_1 = P_0 + h_1\rho g$ and the pressure at B is $P_2 = P_0 + h_2\rho g$, where h_1 and h_2 are the depths of A and B from the free surface. Substituting in (8),

$$h_1\rho g - h_2\rho g = l\rho a_0 \text{ or } \frac{h_1 - h_2}{l} = \frac{a_0}{g} \text{ or } \tan\theta = \frac{a_0}{g}$$

where θ is the inclination of the free surface with the horizontal.

5.3 Steady and Turbulent Flow

Consider a liquid passing through a glass tube (figure 10). Concentrate on a particular point A in the tube and look at the particles arriving at A. If the velocity of the liquid is small, all the particles which come to A will have same speed and will move in same direction. As a particle goes from A to another point B, its speed and direction may change, but all the particles reaching A will have the same speed at A and all the particles reaching B will have the same speed at B. Also, if one particle passing through A has gone through B, then all the particles passing through A go through B. Such a flow of fluid is called a steady flow.

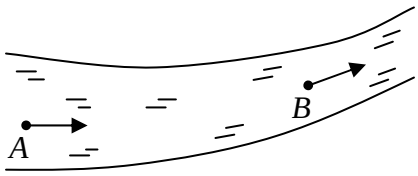


Figure 10

In steady flow the velocity of fluid particles reaching a particular point is the same at all time. Thus each particle follows the same path as taken by a previous particle passing through that point.

If the liquid is pushed in the tube at a rapid rate, the flow may become turbulent. In this case, passing through the same point may be different with and change erratically with time. The motion of water in a high fall or a fast flowing river is, in general, turbulent.

Steady flow is also called streamline flow.

5.3.1 Line of Flow: Streamline

The path taken by a particle in flowing fluid is called its line of flow. The tangent at any point on the line of flow gives the direction of motion of that particle at that point. In the case of steady flow, all the particles passing through a given point follow the same path and hence we have a unique line of flow passing through a given point. In this case, the line of flow is also called a streamline.

5.3.2 Irrotational flow of an incompressible and non-viscous fluid

The analysis of the flow of a fluid becomes much simplified if we consider the fluid to be incompressible and non-viscous and that the flow is irrotational. Incompressibility means that the density of the fluid is same at all the points and remains constant as time passes.

Viscosity of a fluid is related to the internal friction when a layer of fluid slips over another layer. Mechanical energy is lost against such viscous forces. The assumption of a nonviscous fluid will mean that we are neglecting the effect of such internal friction.

Irrotational flow means there is no net angular velocity of fluid particles. When you put some washing powder in a bucket containing water and mix it by rotating your hand in circular path along the wall of the bucket, the water comes into rotational motion.

5.4 Equation of Continuity

We have seen that the fluid going through a tube of flow does not intermix with fluid in other tubes. The total mass of fluid going into the tube through any cross-section should, therefore, be equal to the total mass coming out of the same tube from any other cross-section in the same time. This leads to the equation of continuity.

Let us consider two cross-sections of a tube of flow at the points A and B (figure 11). Let the area of cross-section at A be A_1 and that at B be A_2. Let the speed of the fluid be v_1 at A and v_2 at B.

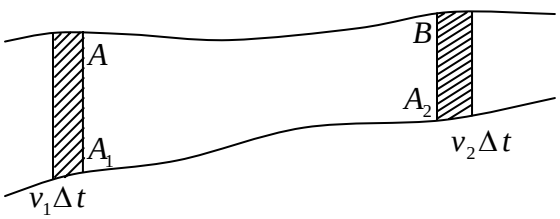


Figure 11

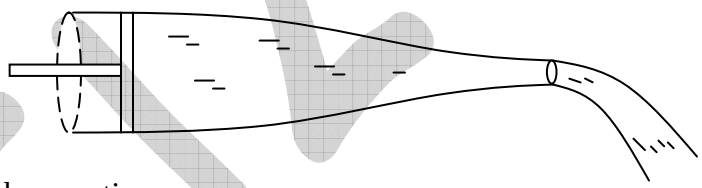
How much fluid goes into the tube through the cross-section at A in a time interval Δt? Let us construct a cylinder of length u at A as shown in the figure. As the fluid at A has speed v all the fluid included in this cylinder will cross through A_1 in the time interval Δt. Thus, the volume of the fluid going into the tube through the cross-section at A is $A_1 v_1 \Delta t$. Similarly, the volume of the fluid going out of the tube through the cross-section at B is $A_2 v_2 \Delta t$ If the fluid is incompressible, we must have

$$A_1 v_1 \Delta t = A_2 v_2 \Delta t$$

or
$$A_1 v_1 = A_2 v_2 \qquad \ldots(9)$$

The product of the area of cross-section and the speed remains the same at all points of a tube of flow. This is called the equation of continuity and expresses the law of conservation of mass in fluid dynamics.

Example: Figure below shows a liquid being pushed out of a tube by pressing a piston. The area of cross-section of the piston is $1.0\,cm^3$ and that of the tube at the outlet is $20\,mm^2$ If the piston is pushed at a speed of $2\,cm/s$ what is the speed of the outgoing liquid?



Solution: From the equation

$$A_1 v_1 = A_2 v_2$$

or
$$(1.0\,cm^2)(2\,cm/s) = (20\,mm^2)v_2$$

or
$$v_2 = \frac{1.0\,cm^2}{20\,mm^2} \times 2\,cm/s$$

$$= \frac{100\,mm^2}{20\,mm^2} \times 2\,cm/s = 10\,cm/s$$

5.5 Bernoulli's Equation

Bernoulli's equation relates the speed of a fluid at a point, the pressure at that point and the height of that point above a reference level. It is just the application of work-energy theorem in the case of fluid flow.

We shall consider the case of irrotational and steady flow of an incompressible and non-viscous liquid. Figure (12) shows such a flow of a liquid in a tube of varying cross-section and varying height. Consider the liquid contained between the cross-sections A and B of the tube. The heights of A and B are h_1 and h_2 respectively from a reference level. This liquid advances into the tube and after a time Δt is contained between the cross-sections A' and B' as shown in figure.

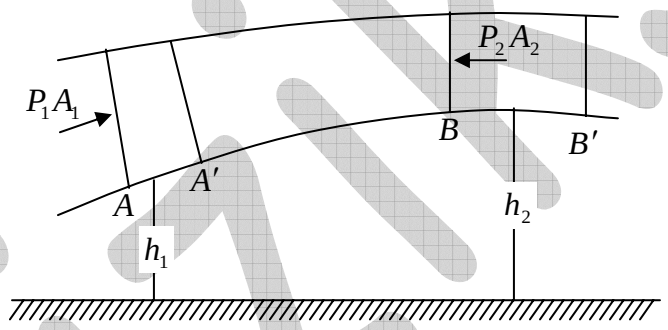


Figure12

Suppose the area of cross-section at $A = A_1$, the area of cross-section at $B = A_2$, the speed of the liquid at $A = v_1$, the speed of the liquid at $B = v_2$, the pressure at $A = P_1$, the pressure at $B = P_2$, and the density of the liquid $= \rho$.

The distance $AA' = v_1\Delta t$ and the distance $BB' = v_2\Delta t$ The volume between A and A' is $A_1v_1\Delta t$ and the volume between B and B' is $A_2v_2\Delta t$ By the equation of continuity,

$$A_1v_1\Delta t = A_2v_2\Delta t$$

The mass of this volume of liquid is

$$\Delta m = \rho A_1v_1\Delta t = \rho A_2v_2\Delta t \qquad \ldots(i)$$

Let us calculate the total work done on the part of the liquid just considered.

The forces acting on this part of the liquid are

(a) P_1A_1 by the liquid on the left

(b) P_2A_2 by the liquid on the right

(c) $(\Delta m)g$, weight of the liquid

(d) f_C, contact forces by the walls of the tube.

In time Δt, the point of application of P_1A_1 is displaced by $AA' = v_1\Delta t$ Thus, the work done by P_1A_1 in time Δt is $W_1 = (P_1A_1)(v_1\Delta t) = P_1\left(\frac{\Delta m}{\rho}\right)$

similarly, the work done by P_2A_2 in time Δt is $W_2 = -(P_2A_2)(v_2\Delta t) = -P_2\left(\frac{\Delta m}{\rho}\right)$

The work done by the weight is equal to the negative of the change in gravitational potential energy.

Change in potential energy $(P.E)$ in time Δt is

$$= P.E. \text{ of } A'BB' - P.E \text{ of } AA'B$$
$$= P.E\ A'B + P.E \text{ of } BB' - P.E \text{ of } AA' - P.E \text{ of } A'B$$
$$= P.E \text{ of } BB' - P.E \text{ of } AA' = (\Delta m)gh_2 - (\Delta m)gh_1$$

Thus the work done by the weight in time Δt is $W_3 = (\Delta m)gh_1 - (\Delta m)gh_2$.

The contact force f_C does no work on the liquid because it is perpendicular to the velocity.

The total work done on the liquid , in the time interval Δt, is $W = W_1 + W_2 + W_3$

$$= P_1\left(\frac{\Delta m}{\rho}\right) - P_2\left(\frac{\Delta m}{\rho} + (\Delta m)gh_1 - (\Delta m)gh_2\right) \qquad \ldots(ii)$$

The change in kinetic energy $(K.E.)$ of the same liquid in time Δt is

$$= K.E. \text{ of } A'BB' - K.E. \text{ of } AA'B$$
$$= K.E. \text{ of } A'B + K.E. \text{ of } BB' - K.E. \text{ of } AA' - K.E. \text{ of } A'B$$
$$= K.E. \text{ of } BB' - K.E. \text{ of } AA' = \frac{1}{2}(\Delta m)v_2^2 - \frac{1}{2}(\Delta m)v_1^2 \qquad \ldots(iii)$$

Since the flow is assumed to be steady, the speed at any point remains constant in time and hence the $K.E$ .of the part $A'B$ is same at initial and final time and cancels out when change in kinetic energy of the system is considered.

By the work-energy theorem, the total work done on the system is equal to the change in its kinetic e Thus,

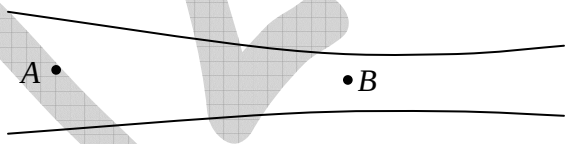
$$P_1\left(\frac{\Delta m}{\rho}\right)-P_2\left(\frac{\Delta m}{\rho}\right)+(\Delta m)gh_1-(\Delta m)gh_2=\frac{1}{2}(\Delta m)v_2^2-\frac{1}{2}(\Delta m)v_1^2$$

$$\frac{P_1}{\rho}+gh_1+\frac{1}{2}v_1^2=\frac{P_2}{\rho}+gh_2+\frac{1}{2}v_2^2 \qquad \Rightarrow P_1+\rho gh_1+\frac{1}{2}\rho v_1^2=P_2+\rho gh_2+\frac{1}{2}\rho v_2^2 \qquad \ldots(10)$$

or $$P+\rho gh+\frac{1}{2}\rho v^2=\text{constant} \qquad \ldots(11)$$

This is known as Bernoulli's equation.

Example: Figure below shows a liquid of density $1200\,kg/m^3$ flowing steadily in a tube of varying cross-section. The cross-section at a point A is $1.0\,cm^2$ and that at B is $20\,mm^2$, the points A and B are in the same horizontal plane. The speed of the liquid at A is $10\,cm/s$ Calculate the difference in pressures at A and B.



Solution: From equation of continuity, the speed v_2 at B is given by,

$$A_1v_1=A_2v_2 \text{ or } (1.0\,cm^2)(10\,cm/s)=(20\,mm^2)v_2 \Rightarrow v_2=\frac{1.0\ \ cm^2}{20\ \ mm^2}\times 10\ \ cm/s=50\ \ cm/s$$

By Bernoulli's equation, $P_1+\rho gh_1+\frac{1}{2}\rho v_1^2-P_2+\rho gh_2+\frac{1}{2}\rho v_2^2$.

$$P_1-P_2=\frac{1}{2}\rho v_2^2-\frac{1}{2}\rho v_1^2=\frac{1}{2}\times(1200kg/m^2)(2500cm^2/s^2-100cm^2/s^2) \qquad \because h_1=h_2$$

$$=600kg/m^2\times 2400\,cm^2/s^2=144\,Pa.$$

Applications of Bernoulli's Equation

(a) Hydrostatics

If the speed of the fluid is zero everywhere, we get the situation of hydrostatics. Putting $v_1 = v_2 = 0$ in the Bernaulli's equation (10),

$$P_1 + \rho g h_1 = P_2 + \rho g h_2 \quad \text{or} \quad P_1 - P_2 = \rho g(h_2 - h_1)$$

as expected from hydrostatics.

(b) Speed of Efflux

Consider a liquid of density ρ filled in a tank of large cross-sectional area A_1. There is a hole of cross-sectional area A_2 at the bottom and the liquid flows out of the tank through the hole. The situation is shown in figure (13). Suppose $A_2 << A_1$.

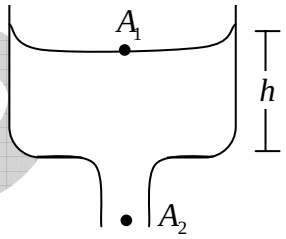


Figure 13

Let v_1 and v_2 be the speeds of the liquid at A_1 and A_2. As both the cross-sections are open to the atmosphere, the pressure there equals the atmospheric pressure P_0. If the height of the free surface above the hole is h,

Bernaulli's equation gives $$P_0 + \frac{1}{2}\rho v_1^2 + \rho g h = P_0 + \frac{1}{2}v_2^2 \qquad \ldots. \text{(i)}$$

By the equation of continuity $A_1 v_1 = A_2 v_2$

Putting v_1 in terms of v_2 in (i), $\frac{1}{2}\rho\left(\frac{A_2}{A_1}\right)^2 v_2^2 + \rho g h = \frac{1}{2}\rho v_2^2 \Rightarrow \left[1 - \left(\frac{A_2}{A_1}\right)^2\right] v_2^2 = 2gh$

If $A_2 << A_1$, this equation reduces to $v_2^2 = 2gh$

or $$v_2 = \sqrt{2gh}$$

The speed of liquid coming out through a hole at a depth h below the free surface is the same as that of a particle fallen freely through the height h under gravity. This is known as Torricelli's theorem. The speed of the liquid coming out is called the speed of efflux.

Example: A water tank is constructed on the top of a building. With what speed will the water come out of a tap 6.0 m below the water level in the tank? Assume steady flow and that the pressure above the water level is equal to the atmospheric pressure.

Solution: The velocity is given by Torricelli's theorem

$$v = \sqrt{2gh} \Rightarrow v = \sqrt{2\times\left(9.8m/s^2\right)\times\left(6.0m\right)} = 11 \;\; m/s$$

Ventury Tube

A ventury tube is used to measure the flow speed of a fluid in a tube. It consists of a constriction or a throat in the tube. As the fluid passes through the constriction, its speed increases in accordance with the equation of continuity. The pressure thus decreases as required by Bernaulli's equation.

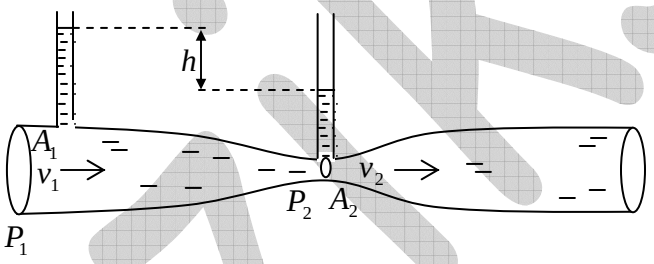


Figure 14

Figure (14) shows a ventury tube through which a liquid of density ρ is flowing. The area of cross-section is A_1 at the wider part and A_2 at the constriction. Let the speeds of the liquid at A_1 and A_2 be v_1 and v_2 and the pressures at A_1 and A_2 be P_1 and P_2 respectively. By the equation of continuity $A_1v_1 = A_2v_2$ ….(i)

and by Bernaulli's equation, $P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2 \Rightarrow \left(P_1 - P_2\right) = \frac{1}{2}\rho\left(v_2^2 - v_1^2\right)$ ….(ii)

Figure (18) also shows two vertical tubes connected to the ventury tube at A_1 and A_2. If the difference in heights of the liquid levels in these tubes is h, we have

$$P_1 - P_2 = \rho g h \quad \Rightarrow 2gh = v_2^2 - v_1^2 \qquad \text{….(iii)}$$

Knowing A_1 and A_2 one can solve equations (i) and (iii) so as to get v_1 and v_2. This allows one to know the rate of flow of liquid past a cross-section.

(d) Aspirator Pump

When a fluid passes through a region at a large speed, the pressure there decreases. This fact finds a number of useful applications. In an aspirator pump a barrel A terminates in a small constriction B (figure 15). The constriction is connected to a vessel containing the liquid to be sprayed through a narrow tube C. The air in the barrel A is pushed by the operator through a piston. As the air passes through the constriction B, its speed is considerably increased and consequently the pressure drops. Due to reduced pressure in the constriction B, the liquid is raised from the vessel and is sprayed with the expelled air.

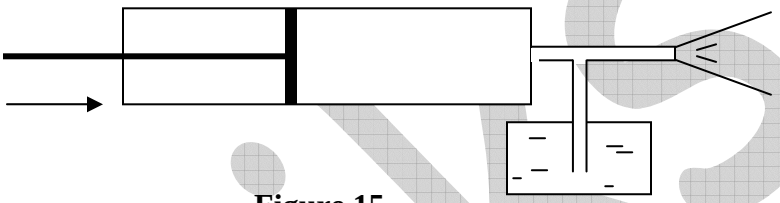


Figure 15

(e) Change of Plane of Motion of a Spinning Ball

Quite often when swing bowlers of cricket deliver the ball, the ball changes its plane of motion in air. Such a deflection from the plane of projection may be explained on the basis of Bernoulli's equation.

Suppose a ball spinning about the vertical direction is going ahead with some velocity in the horizontal direction in otherwise still air. Let us work in a frame in which the centre of the ball is at rest. In this frame the air moves past the ball at a speed v in the opposite direction. The situation is shown (figure 16).

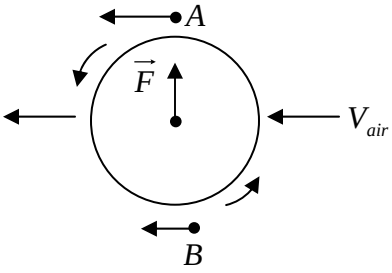


Figure 16

The plane of the figure represents horizontal plane. The air that goes from the A side of the ball in the figure is dragged by the spin of the ball and its speed increases. The air that goes from the B side of the ball in the figure suffers an opposite drag. and its speed decreases. The pressure of air is reduced on the A side and is increased on the B side as required by the Bernoulli's theorem. As a result, a net force F acts on the ball from the B side to the A side due to this pressure difference. This force causes the deviation of the plane of motion.

Example: A beaker of circular cross-section of radius 4 cm is filled with mercury upto a height of 10 cm. Find the force exerted by the mercury on the bottom of the beaker. The atmospheric pressure=$10^5\,N/m^2$, Density of mercury$=13600\,kg/m^3$. Take $g=10\,m/s^2$.

Solution: The pressure at the surface = atmospheric pressure= $10^5\,N/m^2$

The pressure at the bottom = $10^5\,N/m^2+h\rho g=10^5\,N/m^2+(0.1m)\left(13600\frac{kg}{m^2}\right)\left(10\frac{m}{s^2}\right)$

$=10^5\,N/m^2+13600\,N/m^2 \quad =1.136\times10^5\,N/m^2$

Force exerted by the mercury on the bottom

$=\left(1.136\times10^5\ N/m^2\right)\times(3.14\times0.04\,m\times0.04\,m)=571N$

Example: The density of air near earth's surface is $1.3kg/m^2$ and the atmospheric pressure is $1.0\times10^5\,N/m^2$. If the atmosphere had uniform density, same as that observed at the surface of the earth, what would be the height of the atmosphere to exert the same pressure?

Solution: The pressure at the surface of the earth would be

$$p=\rho gh\Rightarrow h=\frac{1.0\times10^6\,N/m^2}{\left(1.3kg/m^3\right)\left(9.8m/s^2\right)}=7850m$$

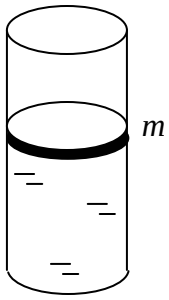
Even Mount Everest (8848 m) would have been outside the atmosphere.

Example: A cylindrical vessel containing a liquid is closed by a smooth piston of mass m as shown in the figure. The area of cross-section of the piston is A. If the atmospheric pressure is P_0 find the pressure of the liquid just below the piston.

Solution: Let the pressure of the liquid just below the piston be P. The forces acting on the piston are

(a) its weight, mg (downward) (b) force due to the air above it, P_0A (downward)

(c) force due to the liquid below it, PA (upward).



If the piston is in equilibrium, $PA=P_0A+mg\Rightarrow P=P_0+\frac{mg}{A}$.

Example: A copper piece of mass $10g$ is suspended by a vertical spring. The spring elongates $1cm$ over its natural length to keep the piece in equilibrium. A beaker containing water is now placed below the piece so as to immerse the piece completely in water. Find the elongation of the spring Density of copper $9000\,kg/m^3$. Take $g = 10m/s^2$

Solution: Let the spring constant be k. When the piece is hanging in air, the equilibrium condition gives

$$k(1cm) = (0.01kg)(10m/s^2)$$

or $$k(1cm) = 0.1N. \qquad \ldots(i)$$

The volume of the copper piece

$$= \frac{0.01kg}{9000kg/m^3} = \frac{1}{9}\times 10^{-5} m^3$$

This is also the volume of water displaced when the piece is immersed in water. The force of buoyancy

= weight of the liquid displaced

$$= \frac{1}{9}\times 10^{-5} m^3 \times (1000kg/m^3)\times(10m/s^2)$$

$$= 0.011N$$

If the elongation of the spring is x when the piece is immersed in water, the equilibrium condition of the piece gives,

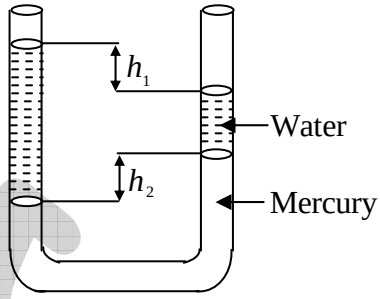
$$kx = 0.1N - 0.011N = 0.089N \qquad \ldots(ii)$$

By (i) and (ii)

$$x = \frac{0.089}{0.1} cm = 0.89cm.$$

MCQ (Multiple Choice Questions)

Q1. A U-tube of uniform cross-sectional area, open to the atmosphere, is partially filled with mercury. Water is then poured into both arms. If the equilibrium configuration of the tube is as shown in figure, with $h_2 = 1.00\ cm$, The value of h_1 is



(a) 130cm

(b) 126 cm

(c) 122 cm

(d) 118 cm

Q2. A cube of wood having an edge dimension of $20.0\ cm$ and a density of $650\ kg/m^3$ floats on water. What is the distance from the horizontal top surface of the cube to the water level?

(a) 6 cm (b) 7 cm (c) 8 cm (d) 9cm

Q3. A cube of wood having an edge dimension of $20.0\ cm$ and a density of $650\ kg/m^3$ floats on water. How much lead weight has to be placed on top of the cube so that its top is just level with the water?

(a) 1.8 kg (b) 1.2 kg (c) 2.4 kg (d) 2.8 kg

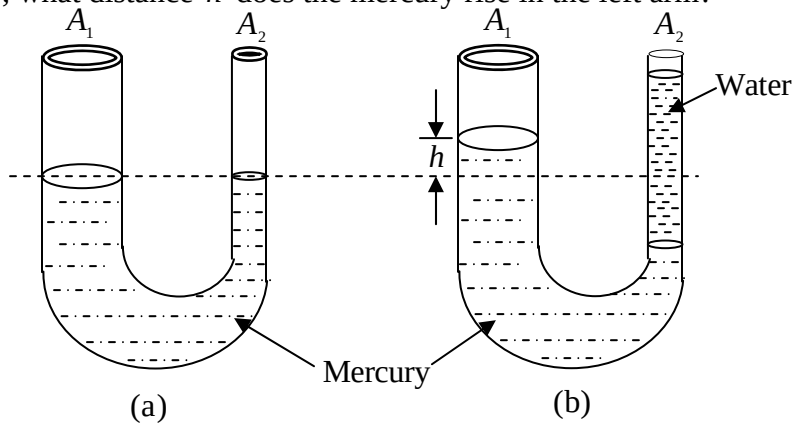
Q4. Mercury is poured into a U-tube as in figure. The left arm of the tube has cross-sectional area A_1 of $10.0\ cm^2$ and the right arm has a cross-sectional area A_2 of $5.00\ cm^2$. One hundred grams of water are then poured into the right arm as in figure .Given that the density of mercury is $13.6\ g/cm^3$, what distance h does the mercury rise in the left arm?

(a) 0.49 cm

(b) 0.32 cm

(c) 0.63 cm

(d) 0.41 cm



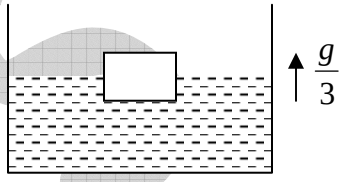
Q5. A cylindrical object of diameter $10\,cm$, height $20\,cm$ and density $8000\,kg/m^3$ is supported by a vertical spring and is half dipped in water. The elongation of the spring in equilibrium condition is

(a) 21 cm (b) 22 cm (c) 23 cm (d) 24 cm

Q6. The height of a mercury barometer is $75\,cm$ at sea level and $50\,cm$ at the top of a hill. Ratio of density of mercury to that air is 104. The height of the hill is

(a) $250\,m$ (b) $2.5\,km$ (c) $1.25\,km$ (d) $750\,m$

Q7. A cubical block is floating in a liquid with half of its volume immersed in the liquid, where the whole system accelerates upward with acceleration of $\frac{g}{3}$, the fraction of volume immersed in the liquid will be



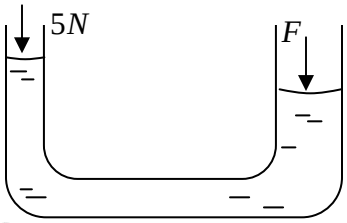
(a) $\frac{1}{2}$ (b) $\frac{3}{8}$ (c) $\frac{2}{3}$ (d) $\frac{3}{4}$

Q8. A vessel contains oil (density $= 0.8\,g/cm^3$ mercury (density $13.6\,g/cm^3$ A homogeneous sphere floats with half its volume immersed in mercury and the other half in oil. The density of the material of the sphere in g/cm^3 is

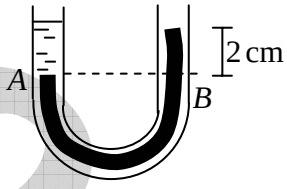
(a) 3.3 (b) 6.4 (c) 7.2 (d) 12.8

NAT (Numerical Answer Type)

Q9. The area of cross-section of the two arms of a hydraulic press are $1cm^2$ and $10cm^2$ respectively (figure). A force of $5N$ is applied on the water in the thinner arm. The Force should be applied on the water in the thicker arm so that the water may remain in equilibrium is_____ N



Q10. The liquids shown in figure in the two arms are mercury specific gravity $=13.6$ and water. If the difference of heights of the mercury columns is $2cm$, the height h of the water column is _____ cm



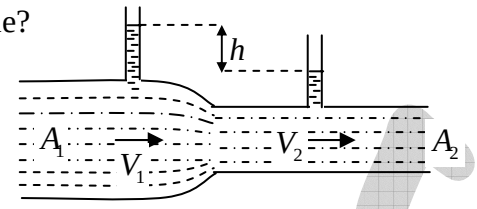
Q11. A cubical block of wood of edges $3cm$ float in water. The lower surface of the cube just touches the free end of a vertical spring fixed at the bottom of the pot. The maximum weight that can be put on the block without wetting it is _____ N (density of wood = $800\,kg/m^3$ and spring constant $=50\,N/m$)

Q12. A spherical aluminum ball of mass $1.26\,kg$ contains an empty spherical cavity that is concentric with the ball. The ball just barely floats in water. The radius of the cavity is _____ cm

Q13. A cylinder of height $20\,m$ is completely filled with water. The velocity of efflux of water (in m/s) through a small hole on the side wall of the cylinder near its bottom is _____.

MSQ (Multiple Select Questions)

Q14. A liquid is flowing through a horizontal tube. The velocities of the liquid in the two sections, which have areas of cross-section A_1 and A_2, the volumes are V_1 and V_2 respectively. The difference in the levels of the two liquid in the two vertical tubes is h. Which of the statements are true?



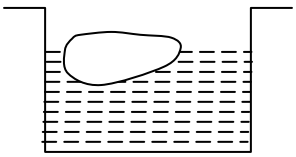
(a) The volume of the liquid flowing through the tube in unit time is A_1V_1

(b) $V_2 - V_1 = \sqrt{2gh}$

(c) $V_2^2 - V_1^2 = 2gh$

(d) The total energy per unit mass of the liquid is the same in both sections of the tube.

Q15. A body floats in a liquid contained in a beaker. The whole system as shown in figure falls freely under gravity. Which of the following can not be upthrust on the body due to liquid?



(a) zero

(b) equal to the weight of the liquid displaced

(c) equal to the weight of the body in air

(d) equal to the weight of the immersed portion of the body.

Q16. Water from a tap emerges vertically downwards with an initial speed of $1.0\ ms^{-1}$. The cross-sectional area of The tap is$10^{-4}\ m^2$. In Assume that the pressure is constant throughout the stream of water, and that the flow is steady. The stream is $0.15\ m$ below the tap, which of the following statements

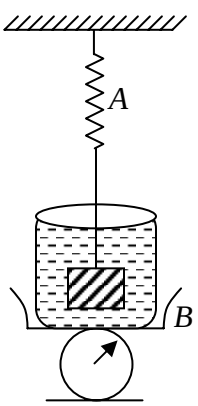
(a) velocity of water stream at $0.15m$ below the tap is $4m/s$

(b) velocity of water stream at $0.15m$ below the tap is $2m/s$

(c) the cross-sectional area of the stream $0.15m$ below the tap is $5\times10^{-5}\ m^2$

(d) the cross-sectional area of the stream $0.15m$ below the tap is $2\times10^{-5}\ m^2$

Q17. The spring balance 4 reads 2 kg with a block in suspended from it. KbalanoeBreads5kgwhen.a beaker with liquid is put on the pan of the balance. The two balances are now so arranged that the hanging mass is inside the liquid in the beaker as shown in the figure. In this situation:



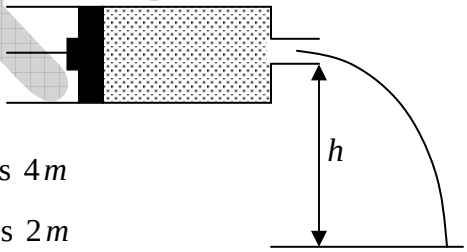
(a) the balance A will read more than $2\ kg$

(b) the balance B will read more than $5\ kg$

(c) the balance A will read less than $2\ kg$ and B will read more than $5\ kg$

(d) the balance A and B will read $2\ kg$ and $5\ kg$ respectively

Q18. A tube has two area of cross-sections as shown in figure. The diameters of the tube are $8\ mm$ and $2\ mm$. The piston is moving with a constant velocity of $0.25\ m/s$, $h = 1.25\ m\left(g = 10\ m/s^2\right)$, which of the following statements are correct?



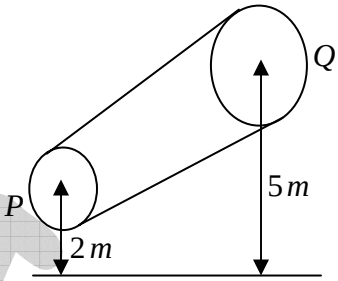
(a) The velocity in the horizontal direction is $4\,m/s$

(b) The velocity in the horizontal direction is $2\,m/s$

(c) The range of water falling on the horizontal surface is $4\,m$

(d) The range of water falling on the horizontal surface is $2\,m$

Q19. A non-viscous liquid of constant density 1000 kg/m^3 in a streamline motion along a tube of variable cross-section. The tube is kept inclined in the vertical plane as shown in figure. The area of cross-section of the tube at two points P and Q at heights of $2m$ and $5m$ are respectively $4\times10^{-3}\ m^2$ and $8\times10^{-3}\ m^2$. The velocity of the liquid at point P is $1\,m/s$. Which of the following statements are correct for work done per unit volume by the pressure and the gravity forces as fluid flow from point P to Q?



(a) Work done per unit volume by the pressure is $29.03\times10^3\ J/m^3$.

(b) Work done per unit volume by the pressure is $29.03\times10^4\ J/m^3$.

(c) Work done per unit volume by gravity is $29.4\times10^3\ J/m^3$.

(d) Work done per unit volume by gravity is $29.4\times10^4\ J/m^3$.

Q20. A large open container of negligible mass and uniform cross-sectional area A has a small hole of cross-sectional area $\frac{A}{100}$ in its side wall near the bottom. The container is kept on a smooth horizontal floor and contains a liquid of density ρ and mass m_0. Assuming that the liquid starts flowing out horizontally through the hole at $t=0$ (where $A=200\,cm^2, \rho=1000\,kg/m^3$ and $m_0=1000\,kg$). Which of the following statements are correct?

(a) the acceleration of the container is $0.4\,m/s^2$

(b) the force is exerted on the container is $200\,N$

(c) fluid velocity when 75% of the liquid has drained out is $15.8\,m/s$

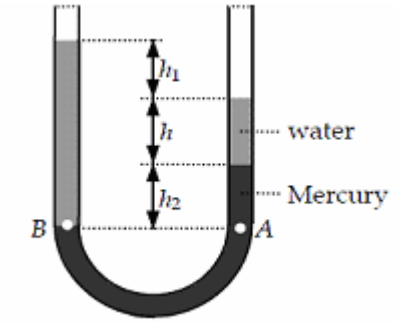
(d) the acceleration of the container is $0.2\,m/s^2$

Solutions

MCQ (Multiple Choice Questions)

Ans. 1: (b)

Solution: Let h be the height of the water column added to the right side of the U-tube. Then when equilibrium is reached, the situation is as shown in the sketch at right. Now consider two points, A and B shown in the sketch, at the level of the water-mercury interface. By Pascal's Principle, the absolute pressure at B is the same as that at A. But,



$$P_A = P_0 + \rho_w gh + \rho_{Hg} gh_2 \text{ and}$$

$$P_B = P_0 + \rho_w g\left(h_1 + h + h_2\right)$$

Thus, from $P_A = P_B$, $\rho_w h_1 + \rho_w h + \rho_w h_2 = \rho_w h + \rho_{Hg} h_2$ or

$$h_1 = \left[\frac{\rho_{Hg}}{\rho_w} - 1\right] h_2 = (13.6 - 1)(1.00\,cm) = 126\ cm$$

Ans. 2: (b)

Solution: 3 (a) According to Archimedes,

$$H = \rho_{water} V_{water}\, g = \left(1.00\,g/cm^3\right)\left[20.0 \times 20.0 \times \left(20.0 - h\right)\right] g$$

But $B =$ Weight of block $= mg = \rho_{wood} V_{wood}\, g = \left(0.650\,g/cm^3\right)\left(20.0\ cm\right)^2 g$

$$0.650\left(20.0\right)^3 g = 1.00\left(20.0\right)\left(20.0\right)\left(20.0 - h\right) g$$

$$20.0 - h = 20.0\left(0.650\right) \text{ so } h = 20.0\left(1 - 0.650\right) = 7.00\ cm$$

Ans. 3: (d)

Solution: $B = F_g + Mg$ where M = mass of lead

$$1.00\left(20.0\right)^3 g = 0.650\left(20.0\right)^3 g + Mg$$

$$M = \left(1.00 - 0.650\right)\left(20.0\right)^3 = 0.350\left(20.0\right)^3 = 2800\,g = 2.80\,kg$$

Ans. 4: (a)

Solution: Using the definition of density, we have

$$h_w = \frac{m_{water}}{A_2 \rho_{water}} = \frac{100\,g}{5.00\ cm^2\left(1.00\ g/cm^3\right)} = 20.0\,cm$$

Sketch (b) at the right represents the situation after the water is added. A volume $(A_2 h_2)$ of mercury has been displaced by water in the right tube. The additional volume mercury now in the left tube is $A_1 h$. Since the total volume of mercury has not changed,

$$A_2 h_2 = A_1 h \quad \text{or} \quad h_2 = \frac{A_1}{A_2} h \qquad \text{(i)}$$

At the level of the mercury-water interface in the right tube, we may write the absolute pressure as:

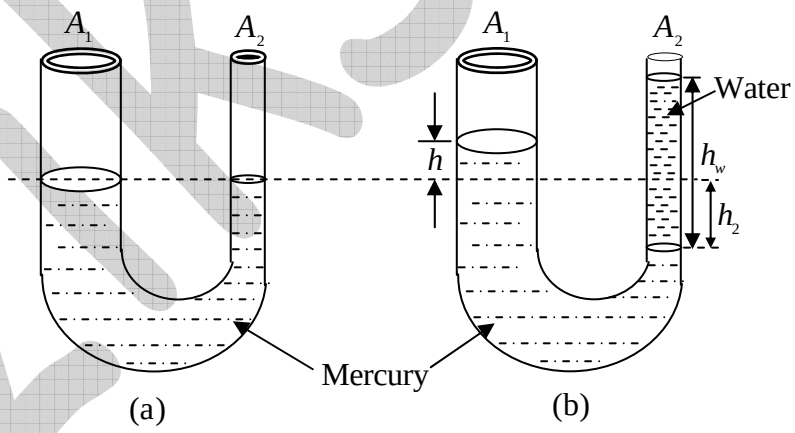
$$P = P_0 + \rho_{water} g h_w$$

The pressure at this same level in the left tube is given by

$$P = P_0 + \rho_{Hg} g\left(h + h_2\right) = P_0 + \rho_{water} g h_w$$

which, using equation (i) above, reduces to

$$\rho_{Hg} h\left[1 + \frac{A_1}{A_2}\right] = \rho_{water} h_w \text{ or } h = \frac{\rho_{water} h_w}{\rho_{Hg}\left(1 + \frac{A_1}{A_2}\right)}$$



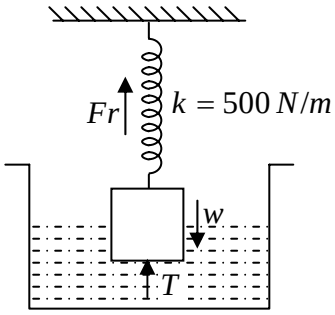
Thus, the level of mercury has risen a distance of

$$h = \frac{\left(1.00\,g/cm^3\right)\left(20.0\ cm\right)}{\left(13.6\,g/cm^3\right)\left(1 + \frac{10.0}{5.00}\right)} = 0.490\ cm \text{ above the original level.}$$

Ans. 5: (d)

Solution: Volume of the cylinder $(V) = \pi r^2 \times H$

$$V = 3.14 \times 5^2 \times 20 = 1571\ cm^3$$



Mass of the cylindrical object $(M) = \rho V$

$$M = \left(8000\ kg/m^3\right)\left(1571 \times 10^{-6}\ m^3\right) = 12.57\,kg$$

When object is half dipped, then the volume displaced in water is V_d = Half volume of cylinder $= 785.4\,cm^3 = 785.4 \times 10^{-6}\ m^3$

In equilibrium conditions; $W = F_r + T \Rightarrow F_r = W - T$

Weight of object $W = \rho V g$

$$W = \left(8000\,kg/m^3\right)\left(1571 \times 10^{-6}\ m^3\right)\left(10\,ms^{-2}\right) = 125.7\ N$$

Force of buoyancy, $T = \rho_w V_d g = \left(1000\ kg/m^3\right)\left(785.4 \times 10^{-6} m^3\right)\left(10\,ms^{-2}\right)$

$$T = 7.85\,N$$

Then $F_r = W - T = 125.7 - 7.85 = 117.85\ N$

Also, $F_r = Kx \Rightarrow x = \dfrac{F_r}{K} = \dfrac{117.85\,N}{500\,N/m} = 0.24\,m$

$$x = 24\ cm$$

Ans. 6: (b)

Solution: Difference of pressure between sea level and the top of the hill

$$\Delta P = \left(h_1 - h_2\right) \times \rho_{Hg} \times g = \left(75 - 50\right) \times 10^{-2} \times \rho_{Hg} \times g$$

and pressure difference due to h meter of air $\Delta P = h \times \rho_{air} \times g$

By equating these two $h \times \rho_{air} \times g = \left(75 - 50\right) \times 10^{-2} \times \rho_{Hg} \times g$

$$\therefore \quad h = 25 \times 10^{-2} \left(\frac{\rho_{Hg}}{\rho_{air}}\right) = 25 \times 10^{-2} \times 10^4 = 2500\,m$$

$\therefore$ Height of the hill $= 2.5\,km$

Ans. 7: (a)

Solution: Fraction of volume immersed in the liquid is

$$V_{in} = \left(\frac{\rho}{\sigma}\right)V$$

where ρ is the density of block and σ is the density of liquid

i.e. it depends upon the densities of the block and liquid. So there will be no change in it.

If system moves upward or downward with constant velocity or acceleration.

Ans. 8: (c)

Solution: weight of floating sphere = Upthrust due to $(Hg + \text{oil})$

$$\therefore \quad V\rho g = \frac{V}{2}\rho\mu g \times g + \frac{V}{2}\rho_{\text{oil}} \times g$$

$$\text{or } \rho = \frac{\rho\mu_g + \rho_{\text{oil}}}{2} = \frac{13.6 + 0.8}{2} = 7.2 \ g\,cm^{-3}$$

$\therefore$ Density of material of sphere $= 7.2\left(g \times cm^{-3}\right)$

NAT (Numerical Answer Type)

Ans. 9: 50

Solution: In equilibrium, the pressures at the two surfaces should be equal as they lie in the same horizontal level. If the atmospheric pressure is P and a force F is applied to maintain the equilibrium, the pressures are

$P_0 + \frac{5N}{1cm^2}$ and $P_0 + \frac{F}{10cm^2}$ respectively. This gives $F = 50N$.

Ans. 10: 27

Solution: Suppose the atmospheric pressure $= P_0$

Pressure at $B = P_0 + (0.02m)\left(13600\frac{kg}{m^3}\right)g$

These pressures are equal as A and B are at the same horizontal level. Thus,

$$h = (0.02m)13.6 = 0.27m = 27cm$$

Ans. 11: 0.36

Solution: Suppose the maximum weight that can be put without wetting it is W. The block in this case is completely immersed in the water. The volume of the displaced water is (V_d)

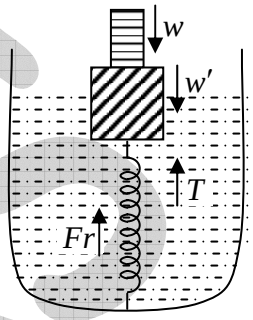
$V_d =$ Volume of block $= a^3 = (3cm)^3$

$V_d = 27\times10^{-6}\ m^3$

The force of byoyancy $T = (\text{mass of displaced liquid})\,g$

$T = V_d \rho_w g = (27\times10^{-6}\ m^3)(1000\,kg/m^3)(9.8\,m)$

$T = 0.27\,N$



Let us calculate the portion of wood dipped in water

Height inside water $= a\times\dfrac{\rho_{wood}}{\rho_{water}} = 3cm\times\dfrac{800}{1000} = 2.4cm$

$\therefore$ Height outside water $= 3cm - 2.4cm = 0.6cm$

The spring is compressed by $0.6cm$ and hence the upward force exerted by the spring is

$F_r = Kx = (50\,N/m)(0.6\times10^{-2}\,m) = 0.3$

$F_r = 0.3\,N$

The force of buoyancy and the spring force taken together balance the weight of the block plus the weight W put on the block.

The weight of block $W' = \rho_{wood}\times V\times g$

$W' = (800\,kg/m^3)(27\times10^{-6}\ m^3)(9.8\,ms^{-2})$

$W' = 0.21\,N$

Thus upward force = downward force

$T + F_r = W + W'$

$0.27 + 0.3 = W + 0.21 \Rightarrow W = 0.36\,N$

Ans. 12: 5.74

Solution: The weight of the ball must be equal to the buoyant force of the water:

$$1.26\,kgg = \rho_{water}\frac{4}{3}\pi r_{outer}^3 g$$

$$r_{outer} = \left(\frac{3\times 1.26\,kg}{4\pi 1000\ kg/m^3}\right)^{1/3} = 6.70\ cm$$

The mass of the ball is determined by the density of aluminum:

$$m = \rho_{A_1} V = \rho_{A_1}\left(\frac{4}{3}\pi r_0^3 - \frac{4}{3}\pi r_i^3\right)$$

$$1.26\,kg = 2700\ kg/m^3\left(\frac{4}{3}\pi\right)\left((0.067\,m)^3 - r_i^3\right)$$

$$1.11\times 10^{-4}\,m^3 = 3.01\times 10^{-4}\,m^3 - r_i^3$$

$$r_i = \left(1.89\times 10^{-4}\,m^3\right)^{1/3} = 5.74\ cm$$

Ans. 13: 20

Solution: The velocity of efflux is

$$V = \sqrt{2gh} = \sqrt{2\times 10\times 10} = 20\,m/s$$

MSQ (Multiple Select Questions)

Ans. 14: (a), (c) and (d)

Solution: By the equation of continuity $A_1V_1 = A_2V_2$

Thus the volume of the liquid flowing through the tube in unit time is same and equal to A_1V_1.

According to Bernoulli's equation which also the consequence of conservation of energy states $P_1 + \frac{1}{2}\rho V_1^2 = P_2 + \frac{1}{2}\rho V_2^2$

and $P_1 - P_2 = \rho gh$

Thus $V_2^2 - V_1^2 = 2gh$

Thus correct statements are (a), (c) and (d).

Ans. 15: (b), (c) and (d)

Solution: Upthrust $= V\rho_{liquid} \times (g - a)$

Where $a =$ downward acceleration

$g =$ acceleration due to gravity

$V =$ volume of the liquid displaced

$\rho_{liquid} =$ density of the liquid

In case of free fall, $a = g$

Thus, upthrust $= V\rho_{liquid}(g - g) = 0$

Ans. 16: (b) and (c)

Solution: By equation of continuity $v_1 A_1 = v_2 A_2$

$v_2 =$ velocity of water stream at $0.15\ m$ below the tap

now $v_2^2 = (1)^2 + 2 \times 10 \times 0.15 = 4$ or $v_2 = 2\ m/s$

$\therefore A_2 = \dfrac{v_1 A_1}{v_2} = \dfrac{1 \times 10^{-4}}{2} = 5 \times 10^{-5}\ m^2$

$\therefore$ cross-section area of stream $= 5 \times 10^{-5}\ m^2$

Ans. 17: (b) and (c)

Solution: When the hanging mass is inside the liquid, the liquid will apply an upthrust on the mass. Hence the balance A will read less than $2\ kg$. Infact m loses a weight when dipped in a liquid.

An upthrust acts on block m

By reaction, an equal force will be exerted on the liquid contained in beaker in the downward direction.

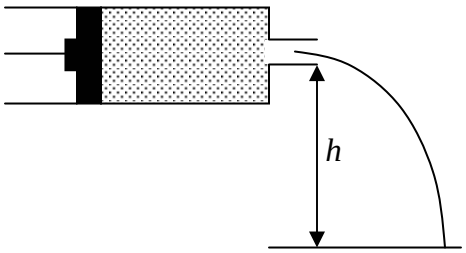
Hence B will read more than $5\ kg$.

Hence (b) and (c) both are correct

Infact (b) is contained in (c). So it is adequate to say that (c) holds good.

Ans. 18: (a) and (d)

Solution: By equation of continuity,



$$A_1 v_1 = A_2 v_2$$

$$A_1 = \pi \times \left(4 \times 10^{-3}\right)^2 m^2 = \pi \times 16 \times 10^{-6}\ m^2$$

$$A_2 = \pi \times \left(1 \times 10^{-3}\right)^2\ m^2 = \pi \times 10^{-6}\ m^2$$

$$v_1 = 0.25\ m/s$$

$$\therefore v_2 = \frac{A_1 v_1}{A_2} \quad \text{or } v_2 = \frac{\left(\pi \times 16 \times 10^{-6}\right) \times (0.25)}{\left(\pi \times 10^{-6}\right)}$$

$$\text{or } v_2 = 16 \times 0.25 = 4\ m/s \qquad \text{(i)}$$

this velocity v_2 is in horizontal direction

vertical height $= h$, time $= t$, Path of water = Parabola

$$\therefore \frac{1}{2} g t^2 = h \quad \text{or } t = \sqrt{\frac{2h}{g}} \qquad \text{(ii)}$$

$\therefore$ Horizontal distance or range $= v_2 t$

$$\therefore \text{Range} = 4 \times \sqrt{\frac{2 \times 1.25}{10}} \quad \text{or Range} = \frac{4 \times 1}{2}$$

$$\text{or Range} = 2\ m$$

Ans. 19: (a) and (c)

Solution: According to equation of continuity,

$$A_1 v_1 = A_2 v_2 \quad \text{or} \quad v_2 = \frac{A_1 v_1}{A_2} = \frac{\left(4 \times 10^{-2}\right) \times (1)}{\left(8 \times 10^{-3}\right)} = \frac{1}{2}$$

$$\therefore v_2 = \frac{1}{2} m/s \qquad \text{(i)}$$

According to Bernoulli's theorem,

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2$$

or $\quad (P_1 - P_2) = \rho g (h_2 - h_1) + \frac{1}{2}\rho\left(v_2^2 - v_1^2\right) \qquad$ (ii)

(i) Work done per unit volume by the pressure:

$$W_1 = P_1 - P_2$$

From equation (ii), we get

$$W_1 = \rho g (h_2 - h_1) + \frac{1}{2}\rho\left(v_2^2 - v_1^2\right).$$ Put v_2 from (i)

$$\therefore W_1 = \left(10^3\right)(9.8)(5-2) + \frac{1}{2}\left(10^3\right)\left(\frac{1}{4} - 1\right)$$

or $\quad W_1 = (29400 - 375) = 29.025 \times 10^3 \ J/m^3$

or $\quad W_1 = 29.025 \times 10^3 \ J/m^3$

$\therefore$ Work done per unit volume by pressure $= 29.025 \times 10^3 \ J/m^3$

(ii) Work done per unit volume by gravity:

$$W_2 = \rho g (h_2 - h_1)$$

or $\quad W_1 = \left(10^3\right)(9.8)(5-2) = 29400 \ J/m^3$

or $\quad W_2 = 29.4 \times 10^3 \ J/m^3$

Work done per unit volume by gravity $= 29.4 \times 10^3 \ J/m^3$

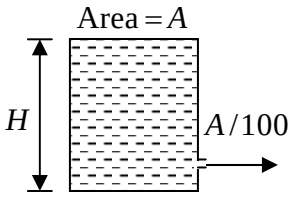
Ans. 20: (b), (c) and (d)

Solution: (i) Acceleration of container:

Mass of water = volume × density

$$\therefore m_0 = (AH)\rho$$

or $\quad H = \frac{m_0}{A\rho}$



$\therefore \quad$ velocity of efflux $= v$

$$\therefore v = \sqrt{gH} = \sqrt{2g \times \frac{m_0}{A\rho}}$$

or $v = \sqrt{\frac{2m_0 g}{A\rho}}$ (i)

When the liquid flows out of the container horizontally, a force is exerted on the container.

Force $F = \rho \times (\text{area of hole}) \times v^2$

or $F = \rho \times \frac{A}{100} \times v^2$

or $F = \frac{\rho A}{100} \times \frac{2m_0 g}{A\rho}$, from (ii)

or $m_0 a = \frac{m_0 g}{50}, (\because F - a_0 a)$ or $a = \frac{g}{50}$

(ii) $F = m_0 a = 1000 \times 0.2 = 200\,N$

(iii) To calculate v when 75% of liquid has drained out:

in this position, $h = \frac{H}{4}$

$\therefore v = \sqrt{2gh}$ or $v = \sqrt{2g\frac{H}{4}}$

or $v = \sqrt{\frac{g \times m_0}{2A\rho}}$ from (i)

or $v = \sqrt{\frac{m_0 g}{2A\rho}} = \sqrt{\frac{1000 \times 10}{2 \times 2 \times 10^{-2} \times 1000}} = \sqrt{250} = 15.8\,m/s$